

Some aspects of stability in thermoelasticity

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Doctor of Philosophy

By

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For my parents my husband, my children and my brothers and sisters.

Abstract

In this thesis we study wave stability in the context of three theories of thermoelasticity: temperature-rate-dependent thermoelasticity TRDTE which was formulated by Green and Lindsay [21]; temperature-rate-dependent thermoelasticity with generalized thermoelasticity, which we label TRDTE + GTE (1), formulated by Chandrasekharaiah and Keshavan [23]; and an alternative theory of temperature-rate-dependent thermoelasticity with generalized thermoelasticity, labelled TRDTE + GTE (2), formulated by Ignaczak [25]. Both anisotropic and isotropic thermoelastic materials are under consideration in this thesis. We are concerned with three cases: unconstrained; the usual deformation-temperature constraint; and the alternative deformation-temperature constraint. We find that in all these cases wave stability/instability is affected by the occurrence of the relaxation times α_0 and α_1 in TRDTE, and α_0 , α_1 and τ in TRDTE + GTE (1) and TRDTE + GTE (2).

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Chapter 1

Introduction

Truesdell and Noll [1] placed the theory of purely mechanical constraints on a firm theoretical basis by postulating that the stress is constitutively determined only to within a reaction stress that does no work in any motion satisfying the constraint. By contrast, there is still disagreement on the general theory of thermomechanical constraints as a restriction on the allowable values of the deformation, temperature and temperature gradient. In the thermomechanical case, Green *et al.* [2] supposed that the stress, entropy and heat flux led to zero entropy production in any process satisfying the thermomechanical constraint. Further work to extend this theory has been undertaken by Gurtin and Podio Guidugli [3]. A similar method was investigated by Andreussi and Podio Guidugli [4], differing only slightly from that of [2, 3], by making the additional assumption of zero energy production by the constraint, although this was criticised by Bertram and Haupt [5] as being very restrictive.

Casey and Krishanaswamy [6] developed an alternative type of theory for deformation-temperature constraints. They obtained expressions for the stress and entropy in the constrained material by considering a related family of unconstrained thermoelastic materials, which itself was obtained by extending the domain of definition of the Helmholtz free energy in a differentiable manner, away from the constraint manifold.

A linearized theory of plane wave propagation in thermoelastic materials was developed by Chadwick [7, 8], and he established that four wave modes are possible in

each direction. Scott [9] proved that each of these modes is stable under quite mild assumptions. We define a stable wave as a wave whose amplitude remains bounded in the direction of propagation. For an isothermal or isentropic elastic material we recall that three stable waves may propagate in each direction and that only two stable waves propagate if a purely mechanical constraint operates, see Chadwick *et al.* [10]. Thus the presence of a constraint removes one mode of propagation but maintains the stability of the system. On physical grounds, this result is to be expected. Consider an initial value problem in an unbounded elastic material. The presence of a constraint implies a connection amongst the initial data thus reducing by one the number of the independent pieces of initial data that must be specified. This, in turn, means that one less mode is required in a Fourier synthesis of the general solution of the initial value problem in a constrained material. Since the solution of the initial value problem is expected to be stable this implies all the modes should be stable as well. Given these outcomes it is at least plausible, therefore, to expect these features of mode-suppression and stability-retention to carry over to the case of an elastic heat conductor that is thermomechanically constrained.

A natural choice of thermomechanical constraint would be one that connects deformation to temperature. This would appear to be well motivated, as physically materials as diverse as vulcanized rubbers and water are considered to be incompressible at uniform temperature, and theoretically there has been a lot of study in the deformation-temperature case. For example, Trapp [11] has examined thermomechanical extensions to inextensibility and incompressibility, whilst Amendola [12] has examined the deformation-temperature constraint more generally. Manacorda [13,14], and independently Beevers [15], have used an approach akin to that of Green *et al.* [2], to consider the propagation of longitudinal waves in an isotropic thermoelastic material that is thermomechanically constrained to be incompressible at uniform temperature. They found, however, that one of the waves is necessarily unstable. More generally, Chadwick and Scott [16] confirmed this conclusion for a fully anisotropic material suffering an arbitrary deformation-temperature constraint, showing that of the four waves that propagate at least one is unstable. These results lead us to the conclusion that the deformation-temperature constraint is unsatisfactory, as no modes are suppressed or

stability maintained when we apply this condition. Scott [17] overcame this problem by postulating a new kind of constraint, which links the deformation with the entropy rather than with temperature. Alts [18] also examined the theory of deformation-temperature constraints but suggested that the undesirable effects of instability may be circumvented by assuming that the constraint only holds approximately.

Chandrasekharaiah [19] presented a broad review of the literature concerned with generalized thermoelasticity theories, including a brief account of the theory of heat conduction with second sound. Müller [20] was the first to introduce the idea of formulating a thermoelasticity theory with second sound. Second sound in heat conduction results from any modification of classical thermoelasticity which renders it hyperbolic rather than parabolic, so that a diffusive mode is changed into a propagating wave mode; this is the same as hyperbolic heat conduction. By considering general constitutive relations for the entropy flux and entropy source, and by making use of a generalized entropy inequality, he developed a precise nonlinear theory of thermoelasticity, which included temperature rate among the constitutive variables and consequently accepted second sound.

Green and Lindsay [21] formulated their own theory of thermoelasticity with second sound which was similar to Müller's. Their theory was clearer and easier to work with than Müller's and is based on an entropy production inequality proposed by Green and Laws [22]. A noteworthy feature of the Green and Lindsay theory is that it retains the classical Fourier law if the material has a centre of symmetry at each point. In the present work we follow Green and Lindsay theory, which we refer to as temperature-rate-dependent-thermoelasticity theory (TRDTE).

Chandrasekharaiah and Keshavan [23] introduced their own theory by combining the field equations of classical thermoelasticity (CTE) and the two models of generalized thermoelasticity. These models are Lord and Shulman theory [24] (GTE) and Green and Lindsay theory [21] (TRDTE). Their method was formed independently of an alternative way to combine TRDTE and GTE which had been given by Ignaczak [25] ten years earlier. Ignaczak formulated this theory in order to encourage further research into this field as there were few exact solutions to the dynamical thermoelasticity

equations at the time.

The classical Fourier's heat conduction law for anisotropic materials is

$$q_i = -k_{ij}\theta_{,j}. \quad (1.1)$$

and the symmetry of k_{ij} , i.e.

$$k_{ij} = k_{ji}, \quad (1.2)$$

is part of the infrastructure of TRDTE theory. In generalized thermoelasticity the heat flux vector $q(\mathbf{x}, t)$ satisfies the following equation, see [19, (4.1)]

$$q_i + \tau \dot{q}_i = -k_{ij}\theta_{,j}, \quad (1.3)$$

in which $q_i(\mathbf{x}, t)$ are the components of the heat flux vector \mathbf{q} and $\theta(\mathbf{x}, t)$ is the temperature increment, both of which are functions of particle position \mathbf{x} and time t . The superposed dot denotes the time derivative and $(\)_{,j}$ denotes the spatial derivative $\partial(\)/\partial x_j$. The quantities k_{ij} are the components of the thermal conductivity tensor \mathbf{k} . In this equation the constant $\tau > 0$ is a relaxation time, which has been used in the system of field equations of temperature-rate-dependent thermoelasticity (TRDTE) combining the generalized thermoelasticity (GTE) theory of Chandrasakharaiiah and Keshavan [23] with that of Ignaczak [25]. It is clear that when $\tau = 0$, equation (1.3) reduces to equation (1.1). The theory derived from the hyperbolic heat conduction equation (1.1) is referred to as Green and Lindsay theory [21]. Straughan [26] has given an excellent account of many theories involving hyperbolic heat conduction; i.e. the propagation of heat waves.

The present work compares the results of the three cases (i) unconstrained, (ii) usual form of deformation-temperature constraint and (iii) alternative form of deformation-temperature constraint, in the context of previous theories of thermoelasticity in separate chapters. Analysis for each theory is performed along the following lines. Solutions of the linearized field equation are sought in the form of plane harmonic waves and the secular equation is found. Low- and high frequency expansions are performed and stability/instability established. In each chapter we considered anisotropic and isotropic thermoelastic materials in two different sections. The linearized equations

for the isotropic case are derived from those given for an anisotropic material [17] by employing convenient isotropic forms and values for the various material constants and tensor components that occur.

In Chapter 2 the theory of temperature-rate-dependent thermoelasticity TRDTE is considered. An unconstrained isotropic thermoelastic material analysis shows that two stable longitudinal waves may propagate in each direction. Through an isotropic thermoelastic material, which is constrained by the usual form of the deformation temperature constraint, one being stable and the other unstable. As frequency varies, these modes occupy parts of a rectangular hyperbola in the complex plane of squared wave speeds. We found mostly similar results when the isotropic thermoelastic materials were constrained by the alternative form of deformation-temperature constraint. By contrast, when the material is anisotropic and unconstrained we found that four finite stable waves propagated in each direction. We encountered difficulty when the material was anisotropic and constrained by both constraints of the usual and alternative forms of deformation-temperature. This is because we could not determine what signs various quantities had in the secular equation and so could not determine how the various eigenvalues interlaced. This forced us to consider a special case which is incompressibility at uniform temperature together with thermal isotropy.

In Chapters 3 and 4 the theories of combining temperature-rate-dependent thermoelasticity TRDTE and generalized thermoelasticity GTE due to Chandrasekharaiah and Keshavan (model 1) and Ignaczak (model 2), respectively, are employed. To do this the field equations are linearized about a uniform equilibrium state and the form of Fourier's law (1.3) is employed which leads to hyperbolic field equations, which is often referred to as the modified or generalized Fourier law, from which the standard law (1.1) is recovered by putting $\tau = 0$ (where τ is the relaxation time). Derivations and analysis performed in Chapters 3 and 4 mirror those performed in Chapter 2. For the unconstrained and isotropic cases in both theories we found that two finite longitudinal waves propagate in each direction, one being stable and the other unstable in the context of TRDTE+GTE (1) and both being unstable in the context of TRDTE+GTE (2). When the material is anisotropic there are up to four stable waves in both theories, three waves being finite and their stability/instability depend-

ing upon the values of the relaxation times τ, α_0 and α_1 , with one mode tending to infinity and being stable in model 1 and unstable in model 2. For an isotropic thermoelastic material which is constrained by the usual or the alternative forms of deformation-temperature constraints in TRDTE+GTE (1) we find that two longitudinal waves may propagate, one being stable and the other unstable, and both finite. But in TRDTE+GTE (2) two longitudinal waves may propagate which are both unstable and finite in the high frequency limit. By contrast, when thermoelastic materials are anisotropic we find that four finite waves may propagate with two of them being unstable in the high frequency case. For the other two waves their stability/instability depend on the values of the constants.

Throughout each of these chapters derivations and analysis have been repeated to make each chapter more self-contained and to improve readability. Many graphical results are presented in each chapter to illustrate various points of the theory.

Chapter 2

Temperature-rate-dependent thermoelasticity (TRDTE)

Introduction

In this chapter we consider temperature-rate-dependent thermoelasticity, which was formulated by Green and Lindsay [21]. The materials under consideration here are anisotropic, and isotropic, which are either unconstrained or constrained by the usual, or alternative, deformation-temperature constraints. The linearized field equations have been given in each case. The stability and instability of waves is affected by the presence of α_0 and α_1 .

2.1 Unconstrained anisotropic TRDTE

2.1.1 Basic equations

We consider a thermoelastic body which possesses a spatially uniform, time-independent, stress-free equilibrium state free of heat flux. For a body with such an equilibrium state the equations of momentum and energy balance in the absence of body force and heat supply, linearized about this equilibrium state, are

$$\sigma_{ij,j} = \rho \ddot{u}_i, \quad -q_{i,i} = \rho T \dot{\phi}, \quad (2.1)$$

respectively, see [8]; where σ_{ij} and q_i are the components of the Cauchy stress tensor and the heat flux vector, respectively. The particle displacement vector $\mathbf{u}(\mathbf{x}, t)$, and the entropy increment $\phi(\mathbf{x}, t)$ are functions of particle position \mathbf{x} and time t . The constant equilibrium values of the density and absolute temperature are denoted by ρ and T , respectively. The notation $(\)_{,j}$ denotes the spatial partial derivative $\partial(\)/\partial x_j$ and the superposed dot denotes the time partial derivative.

The stress, entropy increment and heat flux system of equations is given by Chandrasekharaiah [19, (5.11)–(5.13)]

$$\left. \begin{aligned} \sigma_{ij} &= \tilde{c}_{ijkl} u_{k,l} - \beta_{ij} \left(1 + \alpha_1 \frac{\partial}{\partial t} \right) \theta, \\ \phi &= \rho^{-1} \beta_{ij} u_{i,j} + T^{-1} c \left(1 + \alpha_0 \frac{\partial}{\partial t} \right) \theta - (\rho T)^{-1} c_i \theta_{,i}, \\ q_i &= -k_{ij} \theta_{,j} - c_i \dot{\theta}, \end{aligned} \right\} \quad (2.2)$$

in which α_1, α_0 and c_i are new material constants, where $\alpha_1 \geq \alpha_0 \geq 0$ are relaxation parameters, the isothermal elasticity tensor components at constant equilibrium are \tilde{c}_{ijkl} , the temperature components of stress are β_{ij} , the specific heat at constant deformation is c , k_{ij} are the uniform equilibrium components of the conductivity tensor, and $\theta(x, t)$ is the temperature excess above the equilibrium temperature T . Green and Lindsay [21] show that $\alpha_1 \geq \alpha_0 \geq 0$ is a requirement of the second law of thermodynamics and they observe that if the body has a centre of symmetry at each point then we may take $c_i \equiv 0$; we now make this assumption in common with most work on TRDTE. In order to deduce the field equations of TRDTE for an anisotropic material we need to insert (2.1) into (2.2). In detail, firstly by differentiating (2.2)₁ with respect to x_j we get

$$\sigma_{ij,j} = \tilde{c}_{ijkl} u_{k,lj} - \beta_{ij} \left(1 + \alpha_1 \frac{\partial}{\partial t} \right) \theta_{,j}. \quad (2.2a)$$

Differentiating (2.2)₂ with respect to t we get

$$\dot{\phi} = \rho^{-1} \beta_{ij} \dot{u}_{i,j} + T^{-1} c \left(1 + \alpha_0 \frac{\partial}{\partial t} \right) \dot{\theta}. \quad (2.2b)$$

Multiplying both sides of (2.2b) by ρT we obtain

$$\rho T \dot{\phi} = T \beta_{ij} \dot{u}_{i,j} + \rho c \left(1 + \alpha_0 \frac{\partial}{\partial t} \right) \dot{\theta}, \quad (2.2c)$$

and differentiating (2.2)₃ with respect to x_i gives

$$q_{i,i} = -k_{ij}\theta_{,ij}. \quad (2.2d)$$

Inserting (2.1)₁ into (2.2a), and from (2.2c) and (2.2d) with the aid of (2.1)₂ we get the field equations of TRDTE in the form

$$\left. \begin{aligned} \tilde{c}_{ijkl}u_{k,jl} - \beta_{ij}(\theta + \alpha_1\dot{\theta})_{,j} &= \rho\ddot{u}_i, \\ k_{ij}\theta_{,ij} - T\beta_{ij}\dot{u}_{i,j} &= \rho c(\dot{\theta} + \alpha_0\ddot{\theta}). \end{aligned} \right\} \quad (2.3)$$

These equations form a complete system of field equations for linear TRDTE for a homogeneous and anisotropic material and provide four constant-coefficient, linear partial differential equations for the four unknown functions u_i and θ , see [19, (5.17)–(5.18)]. By setting $\alpha_1 = \alpha_0 = 0$, we recover the field equations of linear classical thermoelasticity (CTE) theory for homogeneous and anisotropic solids, see [30, (2.8a)–(2.8b)].

2.1.2 The secular equation

We are concerned with solutions of equations (2.3) in the form of plane harmonic waves

$$\{u_i, \theta\} = \{U_i, \Theta\} \exp \{i\omega(s\mathbf{n} \cdot \mathbf{x} - t)\}, \quad (2.4)$$

where ω is the angular frequency and \mathbf{n} is the unit wave normal vector in the direction of the propagation, both of which are real constants. The amplitudes $\{U_i, \Theta\}$ and slowness s are in general complex constants. The wave slowness s is the reciprocal of the (complex) wave speed v : $s = 1/v$. We can derive the propagation conditions by inserting (2.4) into (2.3). Firstly, we note the derivatives

$$\begin{aligned} u_{k,lj} &= -(\omega s)^2 n_l n_j U_k e^\chi, \quad (\theta + \alpha_1 \dot{\theta})_{,j} = i\omega s n_j (1 - i\omega \alpha_1) \Theta e^\chi, \quad \ddot{u}_i = -\omega^2 U_i e^\chi, \\ \theta_{,ij} &= -(\omega s)^2 n_i n_j \Theta e^\chi, \quad \dot{u}_{i,j} = \omega^2 s n_j U_i e^\chi, \quad (\dot{\theta} + \alpha_0 \ddot{\theta}) = -i\omega (1 - i\omega \alpha_0) \Theta e^\chi, \end{aligned} \quad (2.4a)$$

where the phase factor χ is defined by

$$\chi = i\omega(s\mathbf{n} \cdot \mathbf{x} - t).$$

Substitute the derivatives (2.4a) into (2.3) and cancel the exponential factor e^χ to give the linear algebraic equations

$$\left. \begin{aligned} (\tilde{c}_{ijkl}n_i n_j - \rho s^{-2} \delta_{ik})U_k + \beta_{ij}n_j \omega^{-1} s^{-1} i(1 - i\omega\alpha_1)\Theta &= 0, \\ T s^{-1} \beta_{ij}n_j U_i + (k_{ij}n_i n_j - i\omega^{-1}(1 - i\omega\alpha_0)c\rho s^{-2})\Theta &= 0. \end{aligned} \right\} \quad (2.5)$$

We now introduce the isothermal and isentropic acoustic tensors and scalar thermal conductivity, respectively,

$$\tilde{Q}_{ij} = \tilde{c}_{ijkl}n_i n_j, \quad \hat{Q}_{ij} = \hat{c}_{ijkl}n_i n_j, \quad k = k_{ij}n_i n_j. \quad (2.6)$$

The isentropic elastic modulus is connected to the isothermal elastic modulus by

$$\hat{c}_{ijkl} := \tilde{c}_{ijkl} + \frac{T}{\rho c} \beta_{ij} \beta_{kl},$$

see [8, (14)]. We can rewrite (2.5) with aid of (2.6) as follows

$$\left. \begin{aligned} (\tilde{Q}_{ij} - \rho s^{-2} \delta_{ik})U_k + \beta_{ij}n_j \omega^{-1} s^{-1} i(1 - i\omega\alpha_1)\Theta &= 0, \\ T s^{-1} \beta_{ij}n_j U_i + (k - i\omega^{-1}(1 - i\omega\alpha_0)c\rho s^{-2})\Theta &= 0. \end{aligned} \right\} \quad (2.7)$$

Rewrite this equation in matrix form to get

$$\begin{bmatrix} \tilde{Q}_{11} - \rho s^{-2} & \tilde{Q}_{12} & \tilde{Q}_{13} & i(1 - i\omega\alpha_1)\omega^{-1} s^{-1} \beta_{1j}n_j \\ \tilde{Q}_{21} & \tilde{Q}_{22} - \rho s^{-2} & \tilde{Q}_{23} & i(1 - i\omega\alpha_1)\omega^{-1} s^{-1} \beta_{2j}n_j \\ \tilde{Q}_{31} & \tilde{Q}_{32} & \tilde{Q}_{33} - \rho s^{-2} & i(1 - i\omega\alpha_1)\omega^{-1} s^{-1} \beta_{3j}n_j \\ T s^{-1} \beta_{1j}n_j & T s^{-1} \beta_{2j}n_j & T s^{-1} \beta_{3j}n_j & k - i\omega^{-1}(1 - i\omega\alpha_0)c\rho s^{-2} \end{bmatrix} \cdot \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ \Theta \end{bmatrix} = 0.$$

These equations have non-zero solutions if and only if the determinant vanishes:

$$\begin{vmatrix} \tilde{Q}_{11} - \rho s^{-2} & \tilde{Q}_{12} & \tilde{Q}_{13} & i(1 - i\omega\alpha_1)\omega^{-1} s^{-1} \beta_{1j}n_j \\ \tilde{Q}_{21} & \tilde{Q}_{22} - \rho s^{-2} & \tilde{Q}_{23} & i(1 - i\omega\alpha_1)\omega^{-1} s^{-1} \beta_{2j}n_j \\ \tilde{Q}_{31} & \tilde{Q}_{32} & \tilde{Q}_{33} - \rho s^{-2} & i(1 - i\omega\alpha_1)\omega^{-1} s^{-1} \beta_{3j}n_j \\ T s^{-1} \beta_{1j}n_j & T s^{-1} \beta_{2j}n_j & T s^{-1} \beta_{3j}n_j & k - i\omega^{-1}(1 - i\omega\alpha_0)c\rho s^{-2} \end{vmatrix} = 0. \quad (2.8)$$

This determinant may be written in the following form giving a version of the secular equation

$$\begin{vmatrix} \tilde{\mathbf{Q}} - w\mathbf{1} & \bar{\alpha}\mathbf{b} \\ \beta\mathbf{b}^T & \gamma_1 \end{vmatrix} = 0, \quad (2.9)$$

where

$$w = \rho s^{-2}, b_i = \beta_{ij} n_j, \bar{\alpha} = i(1 - i\alpha_1 \omega) \omega^{-1} s^{-1}, \beta = T s^{-1}, \text{ and } \gamma_1 = k - i\omega^{-1}(1 - i\omega\alpha_0) c w. \quad (2.10)$$

We can rewrite (2.9) as

$$D \equiv \begin{vmatrix} \tilde{\mathbf{Q}} - w\mathbf{1} & \bar{\alpha}\mathbf{b} + \mathbf{0} \\ \beta\mathbf{b}^T & -\delta + (\gamma_1 + \delta) \end{vmatrix} = 0, \quad (2.11)$$

in which, so far, δ is an arbitrary quantity. Using properties of determinants to expand by the fourth column we have

$$D \equiv \begin{vmatrix} \tilde{\mathbf{Q}} - w\mathbf{1} & \bar{\alpha}\mathbf{b} \\ \beta\mathbf{b}^T & -\delta \end{vmatrix} + \begin{vmatrix} \tilde{\mathbf{Q}} - w\mathbf{1} & \mathbf{0} \\ \beta\mathbf{b}^T & \gamma_1 + \delta \end{vmatrix}.$$

The first determinant written in full is

$$D_1 = \begin{vmatrix} \tilde{Q}_{11} - w & \tilde{Q}_{12} & \tilde{Q}_{13} & \bar{\alpha}b_1 \\ \tilde{Q}_{21} & \tilde{Q}_{22} - w & \tilde{Q}_{23} & \bar{\alpha}b_2 \\ \tilde{Q}_{31} & \tilde{Q}_{32} & \tilde{Q}_{33} - w & \bar{\alpha}b_3 \\ \beta b_1 & \beta b_2 & \beta b_3 & -\delta \end{vmatrix}.$$

To simplify this determinant, remove $\bar{\alpha}\mathbf{b}$ from the fourth column by taking

$$\begin{aligned} &\text{row 1} - \left(\frac{\bar{\alpha}b_1}{-\delta}\right) \text{row 4}, \\ &\text{row 2} - \left(\frac{\bar{\alpha}b_2}{-\delta}\right) \text{row 4}, \\ &\text{row 3} - \left(\frac{\bar{\alpha}b_3}{-\delta}\right) \text{row 4}. \end{aligned}$$

We obtain

$$D_1 = \begin{vmatrix} (\tilde{Q}_{11} - w) - \left(\frac{\bar{\alpha}b_1}{-\delta}\right)\beta b_1 & \tilde{Q}_{12} - \left(\frac{\bar{\alpha}b_1}{-\delta}\right)\beta b_2 & \tilde{Q}_{13} - \left(\frac{\bar{\alpha}b_1}{-\delta}\right)\beta b_3 & 0 \\ \tilde{Q}_{21} - \left(\frac{\bar{\alpha}b_2}{-\delta}\right)\beta b_1 & (\tilde{Q}_{22} - w) - \left(\frac{\bar{\alpha}b_2}{-\delta}\right)\beta b_2 & \tilde{Q}_{23} - \left(\frac{\bar{\alpha}b_2}{-\delta}\right)\beta b_3 & 0 \\ \tilde{Q}_{31} - \left(\frac{\bar{\alpha}b_3}{-\delta}\right)\beta b_1 & \tilde{Q}_{32} - \left(\frac{\bar{\alpha}b_3}{-\delta}\right)\beta b_2 & (\tilde{Q}_{33} - w) - \left(\frac{\bar{\alpha}b_3}{-\delta}\right)\beta b_3 & 0 \\ \beta b_1 & \beta b_2 & \beta b_3 & -\delta \end{vmatrix}.$$

Expanding D_1 by the fourth column leads to

$$D_1 = -\delta \det \left\{ (\tilde{\mathbf{Q}} - w\mathbf{1}) + \frac{\bar{\alpha}\beta}{\delta} \mathbf{b} \otimes \mathbf{b} \right\}. \quad (2.12)$$

and we may determine a relationship between $\tilde{\mathbf{Q}}$ and $\hat{\mathbf{Q}}$ from, see [27, (2.10)],

$$\hat{\mathbf{Q}} = \tilde{\mathbf{Q}} + \frac{T}{\rho c} \mathbf{b} \otimes \mathbf{b}. \quad (2.13)$$

In order to force D_1 to be defined in terms of $\hat{\mathbf{Q}}$ in equation (2.12) we compare with equation (2.13) to obtain

$$\frac{T}{\rho c} = \frac{\bar{\alpha}\beta}{\delta}, \quad (2.14)$$

which fixes the value of δ . Substituting (2.10)_{3,4} into (2.14) we get

$$\delta = i(1 - i\omega\alpha_1)\omega^{-1}cw. \quad (2.15)$$

Thus, the first determinant is given by

$$D_1 = -i(1 - i\omega\alpha_1)\omega^{-1}cw \det\{\hat{\mathbf{Q}} - w\mathbf{1}\}. \quad (2.16)$$

The second determinant of (2.11) is

$$D_2 = \begin{vmatrix} (\tilde{Q}_{11} - w) & \tilde{Q}_{12} & \tilde{Q}_{13} & 0 \\ \tilde{Q}_{21} & (\tilde{Q}_{22} - w) & \tilde{Q}_{23} & 0 \\ \tilde{Q}_{31} & \tilde{Q}_{32} & (\tilde{Q}_{33} - w) & 0 \\ \beta b_1 & \beta b_2 & \beta b_3 & \gamma_1 + \delta \end{vmatrix} = (\gamma_1 + \delta) \det\{\tilde{\mathbf{Q}} - w\mathbf{1}\}. \quad (2.17)$$

So, after inserting (2.10)₅ and (2.15) into (2.17), the second determinant may be written as

$$D_2 = [k - i\omega^{-1}(1 - i\omega\alpha_0)cw + i(1 - i\omega\alpha_1)\omega^{-1}cw] \det\{\tilde{\mathbf{Q}} - w\mathbf{1}\}. \quad (2.18)$$

Therefore, the determinant $D = D_1 + D_2$ becomes

$$\begin{aligned} D &= -i(1 - i\omega\alpha_1)\omega^{-1}wc \det\{\hat{\mathbf{Q}} - w\mathbf{1}\} \\ &\quad + (k - i\omega^{-1}(1 - i\omega\alpha_0)cw + i(1 - i\omega\alpha_1)\omega^{-1}cw) \det\{\tilde{\mathbf{Q}} - w\mathbf{1}\}. \end{aligned} \quad (2.19)$$

Dividing D by $(-i(1 - i\omega\alpha_1)\omega^{-1}c)$, the secular equation (2.9) becomes

$$\begin{aligned} &w \det\{\hat{\mathbf{Q}} - w\mathbf{1}\} \\ &+ \frac{i\omega c^{-1}}{(1 - i\omega\alpha_1)} (k - i\omega^{-1}(1 - i\omega\alpha_0)cw + i(1 - i\omega\alpha_1)\omega^{-1}cw) \det\{\tilde{\mathbf{Q}} - w\mathbf{1}\} = 0. \end{aligned} \quad (2.20)$$

On simplifying this equation we get

$$w \det \{ \hat{\mathbf{Q}} - w \mathbf{1} \} + \left[\frac{i\omega(\alpha_1 - \alpha_0)w + i\omega c^{-1}k}{1 - i\omega\alpha_1} \right] \det \{ \tilde{\mathbf{Q}} - w \mathbf{1} \} = 0. \quad (2.21)$$

This is the secular equation for unconstrained anisotropic TRDTE and has not previously appeared in the literature.

Putting $\alpha_1 = \alpha_0 = 0$ we recover the secular equation of unconstrained anisotropic material in classical thermoelasticity, see [30, (2.17)].

In terms of γ , a constant with the physical dimensions of stress, introduced in order to non-dimensionalize the equations, we define the frequency $\omega^* = \gamma c/k$. This leads to the following non-dimensional forms for the frequency ω , relaxation times α_0 and α_1 , squared wave speed w , and the isentropic and isothermal acoustic tensors $\hat{\mathbf{Q}}$ and $\tilde{\mathbf{Q}}$, respectively:

$$\omega' = \omega/\omega^*, \quad \alpha'_0 = \alpha_0\omega^*, \quad \alpha'_1 = \alpha_1\omega^*, \quad w' = w\gamma^{-1}, \quad \hat{\mathbf{Q}}' = \hat{\mathbf{Q}}\gamma^{-1}, \quad \tilde{\mathbf{Q}}' = \tilde{\mathbf{Q}}\gamma^{-1}. \quad (2.22)$$

By inserting these non-dimensional quantities (2.22) into (2.21) we obtain the dimensionless secular equation

$$w \det \{ w \mathbf{1} - \hat{\mathbf{Q}} \} + \left[\frac{i\omega \{ 1 + w(\alpha_1 - \alpha_0) \}}{1 - i\omega\alpha_1} \right] \det \{ w \mathbf{1} - \tilde{\mathbf{Q}} \} = 0, \quad (2.23)$$

where we have dropped the dashes for convenience. Equation (2.23) is a quartic in the squared wave speed w with coefficients depending on the frequency ω . The roots $w(\omega)$ of (2.23) represent the possible modes of wave propagation which form four branches in the complex w -plane as is shown later in the graphical results. For a wave mode to be linearly stable requires the condition of stability for $0 \leq \omega < \infty$, which is

$$\text{Im } w(\omega) \leq 0, \quad (2.24)$$

see [9, (18)]. So for positive ω , stable branches $w(\omega)$ are those which lie in the lower half of the complex w -plane. Each branch of the secular equation is examined in detail, with low and high frequency expansions being performed, and stability/instability proved for the entire frequency range.

2.1.3 Stability considerations

We recall that $\hat{\mathbf{Q}}$ and $\tilde{\mathbf{Q}}$ are the isentropic and isothermal acoustic tensors, respectively. The quantities $\hat{q}_i, i = 1, 2, 3$, denote the eigenvalues of $\hat{\mathbf{Q}}$, with $\tilde{q}_i, i = 1, 2, 3$, denoting the eigenvalues of $\tilde{\mathbf{Q}}$. The interlacing property

$$0 < \tilde{q}_1 \leq \hat{q}_1 \leq \tilde{q}_2 \leq \hat{q}_2 \leq \tilde{q}_3 \leq \hat{q}_3, \quad (2.25)$$

was demonstrated in [9]. The inequality $\tilde{q}_1 > 0$ follows from the positive definiteness of $\tilde{\mathbf{Q}}$, see [9]. Now we can rewrite the secular equation (2.21) in terms of the eigenvalues, \hat{q}_i and $\tilde{q}_i, i = 1, 2, 3$ as follows

$$w(w - \hat{q}_1)(w - \hat{q}_2)(w - \hat{q}_3) + \left(\frac{i\omega\{1 + w(\alpha_1 - \alpha_0)\}}{1 - i\omega\alpha_1} \right) (w - \tilde{q}_1)(w - \tilde{q}_2)(w - \tilde{q}_3) = 0. \quad (2.26)$$

The secular equation (2.26) can also be written as

$$\hat{F}(w) + \left(\frac{i\omega\{1 + w(\alpha_1 - \alpha_0)\}}{1 - i\omega\alpha_1} \right) \tilde{G}(w) = 0. \quad (2.27)$$

where

$$\hat{F}(w) = w \prod_{i=1}^3 (w - \hat{q}_i), \quad \tilde{G}(w) = \prod_{i=1}^3 (w - \tilde{q}_i). \quad (2.28)$$

Low frequency expansions

When $\omega = 0$, the roots of the secular equation (2.27) are the zeros of $\hat{F}(w)$, namely, $w = \hat{q}_i, i = 0, 1, 2, 3$, defining $\hat{q}_0 \equiv 0$.

Taylor expansions of the roots of the secular equation take the form

$$w_i(\omega) = \hat{q}_i + \sum_{n=1}^{\infty} d_n^{(i)} (-i\omega)^n, \quad i = 0, 1, 2, 3. \quad (2.29)$$

We can identify the first coefficient $d_1^{(i)}, i = 0, 1, 2, 3$, by substituting (2.29) into (2.27), to get

$$d_1^{(i)} = \{1 + \hat{q}_i(\alpha_1 - \alpha_0)\} \frac{\tilde{G}(\hat{q}_i)}{\hat{F}'(\hat{q}_i)}, \quad i = 0, 1, 2, 3. \quad (2.30)$$

When $i = 1$,

$$d_1^{(1)} = \{1 + \hat{q}_1(\alpha_1 - \alpha_0)\} \frac{(\hat{q}_1 - \tilde{q}_1)(\hat{q}_1 - \tilde{q}_2)(\hat{q}_1 - \tilde{q}_3)}{\hat{q}_1(\hat{q}_1 - \hat{q}_2)(\hat{q}_1 - \hat{q}_3)} > 0. \quad (2.31)$$

Thus,

$$w_1 = \hat{q}_1 - i\omega\{1 + \hat{q}_1(\alpha_1 - \alpha_0)\} \frac{\tilde{G}(\hat{q}_1)}{\hat{F}'(\hat{q}_1)} + O(\omega^2). \quad (2.32)$$

Similarly, when $i = 2, 3$, we get

$$w_2 = \hat{q}_2 - i\omega\{1 + \hat{q}_2(\alpha_1 - \alpha_0)\} \frac{\tilde{G}(\hat{q}_2)}{\hat{F}'(\hat{q}_2)} + O(\omega^2),$$

$$w_3 = \hat{q}_3 - i\omega\{1 + \hat{q}_3(\alpha_1 - \alpha_0)\} \frac{\tilde{G}(\hat{q}_3)}{\hat{F}'(\hat{q}_3)} + O(\omega^2).$$

When $i = 0$,

$$d_1^{(0)} = \frac{\tilde{q}_1 \tilde{q}_2 \tilde{q}_3}{\hat{q}_1 \hat{q}_2 \hat{q}_3} > 0, \quad (2.33)$$

so that,

$$w_0 = -i\omega \frac{\tilde{G}(0)}{\hat{F}'(0)} + O(\omega^2). \quad (2.34)$$

It is clear that the condition of stability (2.24), namely, $\text{Im } w_i < 0$, $i = 0, 1, 2, 3$, is confirmed for each branch. Therefore, there are four stable waves in the low frequency limit.

High frequency expansions

When $\omega \rightarrow \infty$, the roots of the secular equation (2.27) are given by the zeros of $H(w)$ where

$$H(w) := w(w - \hat{q}_1)(w - \hat{q}_2)(w - \hat{q}_3) - \frac{1}{\alpha_1} \{1 + w(\alpha_1 - \alpha_0)\} (w - \tilde{q}_1)(w - \tilde{q}_2)(w - \tilde{q}_3). \quad (2.35)$$

It is clear that $H(w)$ is a quartic in w , so there are four zeros. The zeros of $H(w)$ are denoted by $\bar{q}_1, \bar{q}_2, \bar{q}_3, \bar{q}_4$. In order to find these zeros we need to examine the sign changes of $H(w)$. By using the inequalities (2.25) and the relationship (2.35) we find

that

$$\begin{aligned}
H(0) &= -\alpha_1^{-1}(-\tilde{q}_1)(-\tilde{q}_2)(-\tilde{q}_3) > 0, \\
H(\tilde{q}_1) &= \tilde{q}_1(\tilde{q}_1 - \hat{q}_1)(\tilde{q}_1 - \hat{q}_2)(\tilde{q}_1 - \hat{q}_3) < 0, \\
H(\hat{q}_1) &= -\alpha_1^{-1}\{1 + \hat{q}_1(\alpha_1 - \alpha_0)\}(\hat{q}_1 - \tilde{q}_1)(\hat{q}_1 - \tilde{q}_2)(\hat{q}_1 - \tilde{q}_3) < 0, \\
H(\tilde{q}_2) &= \tilde{q}_2(\tilde{q}_2 - \hat{q}_1)(\tilde{q}_2 - \hat{q}_2)(\tilde{q}_2 - \hat{q}_3) > 0, \\
H(\hat{q}_2) &= -\alpha_1^{-1}\{1 + \hat{q}_2(\alpha_1 - \alpha_0)\}(\hat{q}_2 - \tilde{q}_1)(\hat{q}_2 - \tilde{q}_2)(\hat{q}_2 - \tilde{q}_3) > 0, \\
H(\tilde{q}_3) &= \tilde{q}_3(\tilde{q}_3 - \hat{q}_1)(\tilde{q}_3 - \hat{q}_2)(\tilde{q}_3 - \hat{q}_3) < 0, \\
H(\hat{q}_3) &= -\alpha_1^{-1}\{1 + \hat{q}_3(\alpha_1 - \alpha_0)\}(\hat{q}_3 - \tilde{q}_1)(\hat{q}_3 - \tilde{q}_2)(\hat{q}_3 - \tilde{q}_3) < 0, \\
H(\infty) &= \infty > 0.
\end{aligned} \tag{2.36}$$

The inequalities (2.36) may be used to determine the positions of the zeros of $H(w)$, so that \bar{q}_1 is between 0 and \tilde{q}_1 , \bar{q}_2 is between \hat{q}_1 and \tilde{q}_2 , \bar{q}_3 is between \hat{q}_2 and \tilde{q}_3 and \bar{q}_4 is between \hat{q}_3 and ∞ . Therefore, they interlace according to

$$0 < \bar{q}_1 \leq \tilde{q}_1 \leq \hat{q}_1 \leq \bar{q}_2 \leq \tilde{q}_2 \leq \hat{q}_2 \leq \bar{q}_3 \leq \tilde{q}_3 \leq \hat{q}_3 \leq \bar{q}_4. \tag{2.37}$$

Using (2.28) we rewrite the definition (2.35) as

$$\hat{F}(w) = H(w) + \frac{1}{\alpha_1}\{1 + w(\alpha_1 - \alpha_0)\}\tilde{G}(w). \tag{2.38}$$

The secular equation (2.27) can be written as

$$\hat{F}(w) = \left(\frac{-i\omega}{1 - i\omega\alpha_1} \right) \{1 + w(\alpha_1 - \alpha_0)\}\tilde{G}(w). \tag{2.39}$$

By subtracting (2.38) from (2.39) in order to eliminate $\hat{F}(w)$, we obtain the secular equation in the form

$$H(w) + \left(\frac{1}{\alpha_1} + \frac{i\omega}{1 - i\omega\alpha_1} \right) \{1 + w(\alpha_1 - \alpha_0)\}\tilde{G}(w) = 0. \tag{2.40}$$

Define a quartic polynomial

$$\bar{h}(w) = (w - \bar{q}_1)(w - \bar{q}_2)(w - \bar{q}_3)(w - \bar{q}_4),$$

which must be a scalar multiple of $H(w)$ because both have the same four zeros. By comparing coefficients of w^4 we see that

$$H(w) := \frac{\alpha_0}{\alpha_1} \bar{h}(w). \tag{2.41}$$

Inserting (2.41) into (2.40) we obtain the secular equation in the form

$$\alpha_0(1 - i\omega\alpha_1)\bar{h}(w) + \{1 + w(\alpha_1 - \alpha_0)\}\tilde{G}(w) = 0. \quad (2.42)$$

Taylor expansions of the roots of the secular equation in the high frequency limit take the form

$$w_i(\omega) = \bar{q}_i + \sum_{n=1}^{\infty} d_n^{(i)} (i\omega\alpha_1)^{(-n)}, \quad i = 1, 2, 3, 4. \quad (2.43)$$

By substituting (2.43) into (2.42), we get the coefficients of $(i\omega\alpha_1)^{-1}$:

$$d_1^{(i)} = \alpha_0^{-1} \{1 + \bar{q}_i(\alpha_1 - \alpha_0)\} \tilde{G}(\bar{q}_i) / \bar{h}'(\bar{q}_i), \quad i = 0, 1, 2, 3.$$

When $i = 1$, for example,

$$d_1^{(1)} = \alpha_0^{-1} \{1 + \bar{q}_1(\alpha_1 - \alpha_0)\} \frac{(\bar{q}_1 - \tilde{q}_1)(\bar{q}_1 - \tilde{q}_2)(\bar{q}_1 - \tilde{q}_3)}{(\bar{q}_1 - \bar{q}_2)(\bar{q}_1 - \bar{q}_3)(\bar{q}_1 - \bar{q}_4)} > 0. \quad (2.44)$$

Positivity of $d_1^{(1)}$ is guaranteed by the interlacing properties (2.37). Thus,

$$w_1(\omega) = \bar{q}_1 - i(\alpha_1\omega)^{-1} \alpha_0^{-1} \{1 + \bar{q}_1(\alpha_1 - \alpha_0)\} \tilde{G}(\bar{q}_1) / \bar{h}'(\bar{q}_1) + O(\omega^{-2}). \quad (2.45)$$

Similarly, when $i = 2, 3, 4$, we obtain

$$w_2(\omega) = \bar{q}_2 - i(\alpha_1\omega)^{-1} \alpha_0^{-1} \{1 + \bar{q}_2(\alpha_1 - \alpha_0)\} \tilde{G}(\bar{q}_2) / \bar{h}'(\bar{q}_2) + O(\omega^{-2}),$$

$$w_3(\omega) = \bar{q}_3 - i(\alpha_1\omega)^{-1} \alpha_0^{-1} \{1 + \bar{q}_3(\alpha_1 - \alpha_0)\} \tilde{G}(\bar{q}_3) / \bar{h}'(\bar{q}_3) + O(\omega^{-2}),$$

$$w_4(\omega) = \bar{q}_4 - i(\alpha_1\omega)^{-1} \alpha_0^{-1} \{1 + \bar{q}_4(\alpha_1 - \alpha_0)\} \tilde{G}(\bar{q}_4) / \bar{h}'(\bar{q}_4) + O(\omega^{-2}).$$

Since $d_1^{(i)} > 0$, $i = 1, 2, 3, 4$, the stability condition (2.24) is satisfied, so that there are four stable waves in the high frequency limit.

Stability for all frequencies

We have shown that all four branches are stable in the low and high frequency limits. In order to prove the stability of each one throughout the entire frequency range $0 < \omega < \infty$, we must prove that a branch may cut the real axis only at the low and high frequency limits. If this were not true we would be able to solve the secular equation (2.27) for real w and some ω in the range $0 < \omega < \infty$. Rearrange (2.27) into the form

$$\frac{i\omega}{1 - i\omega\alpha_1} = \frac{-\hat{F}(w)}{\{1 + w(\alpha_1 - \alpha_0)\}\tilde{G}(w)}. \quad (2.46)$$

For real w the right-hand side of (2.46) is real and so cannot be equal to the (necessarily) complex left-hand side for any ω in the range $0 < \omega < \infty$. Thus, all branches are stable for all frequencies $0 \leq \omega < \infty$.

Numerical results

Figure 2.1 shows an example for different values of α_1 and α_0 such that $\alpha_1 \geq \alpha_0 \geq 0$, demonstrating the effect of α_1 and α_0 increasing. In each sub-figure we select the same values of $\tilde{q}_i, i = 1, 2, 3$, and $\hat{q}_i, i = 1, 2, 3$. The low frequency limits are marked with a \times and the high frequency limits with a \circ in the first sub-figure, which corresponds to classical thermoelasticity, and with \bullet in the others which correspond to temperature-rate-dependent-thermoelasticity (TRDTE). It is clear that all branches lie in the lower complex w -plane and so satisfy the stability condition (2.24).

As we see in Figure 2.1, part (a) represents classical thermoelasticity (CTE) and all branches are stable but three of them are finite and one an infinite branch of $w(\omega)$ by which we mean a branch such that $w \rightarrow \infty$ as $\omega \rightarrow \infty$. However, the other parts (b)–(f) for temperature-rate-dependent-thermoelasticity (TRDTE), show that the existence of relaxation times α_0 and α_1 maintains stability and makes all branches finite.

In Figure 2.2, in which $\alpha_0 = 0$ and $\alpha_1 > 0$, we see that the four branches are always stable, but three of them are finite and one is infinite, i.e. $w \rightarrow \infty$, as $\omega \rightarrow \infty$. This is because for $\alpha_0 = 0$, $H(w)$ defined by (2.35) becomes cubic in w , rather than quartic, so the the fourth root \bar{h}_4 changes character from a finite positive real value to a large negative imaginary value.

Putting $\alpha_1 = \alpha_0 > 0$ in Figure 2.3, we get four stable finite branches for all frequencies.

The low frequency branch starting from the origin and the high frequency branch ending at infinity in the w -plane are named diffusive modes. On the other hand, other branches that begin and end close to the real axis, are named elastic modes, see Chadwick [7].

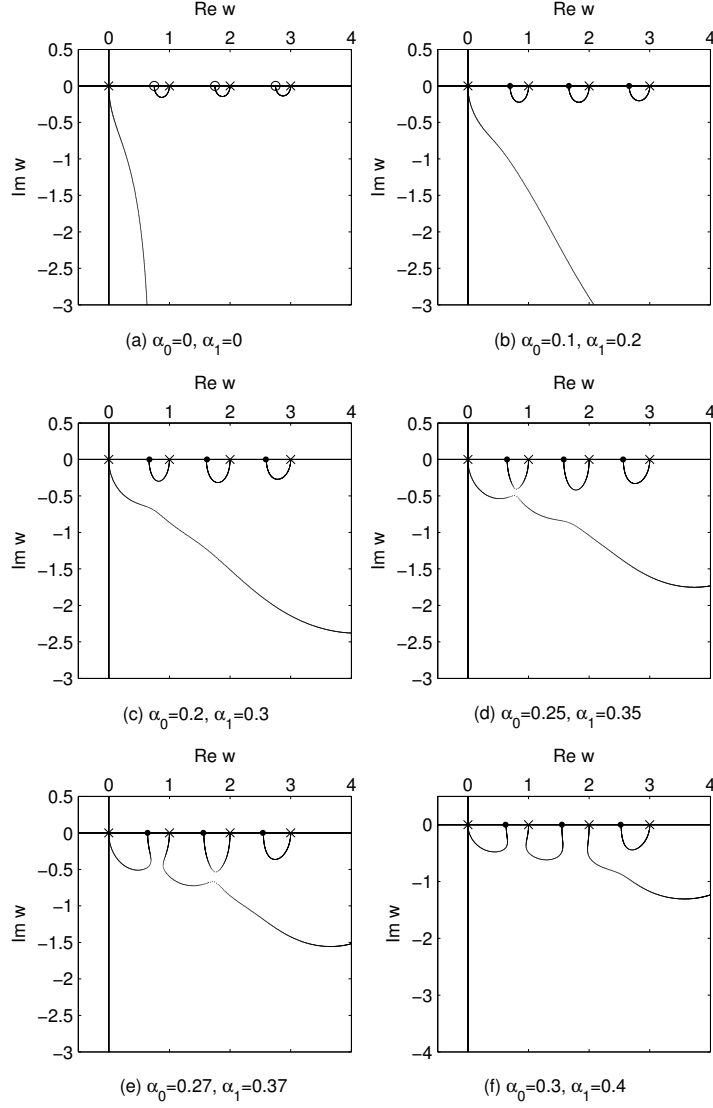


Figure 2.1: The four branches of the secular equation for unconstrained anisotropic thermelasticty material for temperature-rate-dependent thermoelasticity theory. For each part, $\tilde{q}_1 = 0.75$, $\tilde{q}_2 = 1.75$, $\tilde{q}_3 = 2.75$, $\hat{q}_1 = 1$, $\hat{q}_2 = 2$, $\hat{q}_3 = 3$.

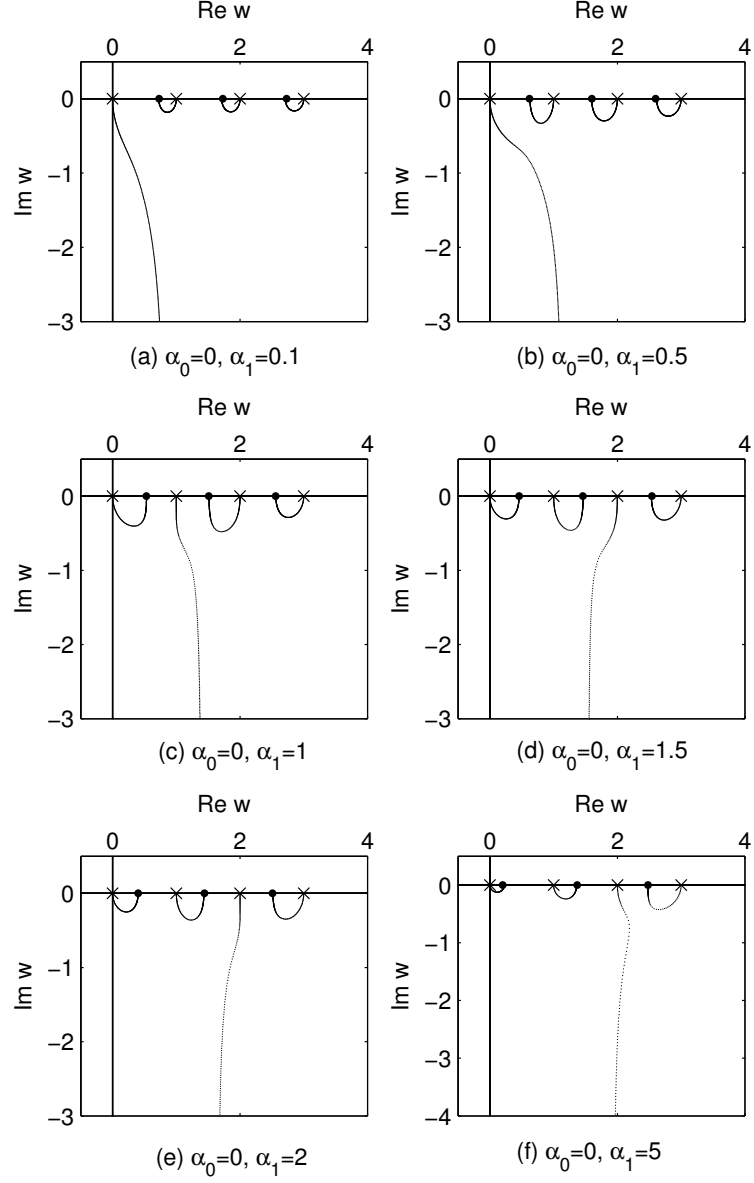


Figure 2.2: The four branches of the secular equation for unconstrained anisotropic thermelastic material for temperature-rate-dependent thermoelasticity theory. For each part, $\tilde{q}_1 = 0.75$, $\tilde{q}_2 = 1.75$, $\tilde{q}_3 = 2.75$, $\hat{q}_1 = 1$, $\hat{q}_2 = 2$, $\hat{q}_3 = 3$.

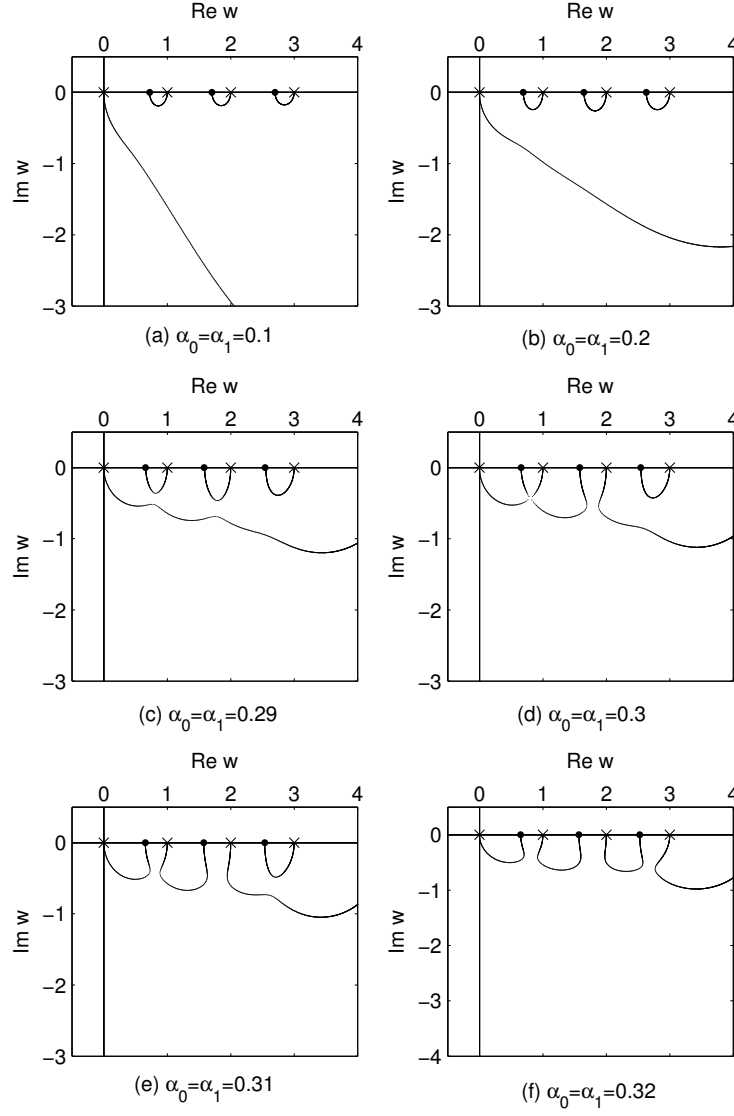


Figure 2.3: The four branches of the secular equation for unconstrained anisotropic thermelastoc material for temperature-rate-dependent thermoelasticity theory. For each part, $\tilde{q}_1 = 0.75$, $\tilde{q}_2 = 1.75$, $\tilde{q}_3 = 2.75$, $\hat{q}_1 = 1$, $\hat{q}_2 = 2$, $\hat{q}_3 = 3$.

2.2 Unconstrained isotropic TRDTE

2.2.1 The field equations

The system of field equations of linear TRDTE for a homogeneous and anisotropic materials is (2.3)

$$\left. \begin{aligned} \tilde{c}_{ijkl}u_{k,jl} - \beta_{ij}(\theta + \alpha_1\dot{\theta})_{,j} &= \rho\ddot{u}_i, \\ k_{ij}\theta_{,ij} - T\beta_{ij}\dot{u}_{i,j} &= \rho c(\dot{\theta} + \alpha_0\ddot{\theta}). \end{aligned} \right\} \quad (2.47)$$

We have defined all symbols earlier in Section 2.1.1. For an isotropic thermoelastic body the components \tilde{c}_{ijkl} , β_{ij} and k_{ij} take the simple forms

$$\tilde{c}_{ijkl} = \tilde{\lambda} \delta_{ij}\delta_{kl} + \tilde{\mu} (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \quad \beta_{ij} = \beta\delta_{ij}, \quad k_{ij} = k\delta_{ij}, \quad (2.48)$$

in which $\tilde{\lambda}$ and $\tilde{\mu}$ are the isothermal Lamé constants and δ_{ij} denote the components of the unit tensor, β is the scalar temperature coefficient of stress and k is the scalar thermal conductivity. Inserting (2.48) into (2.47) gives the field equations for an isotropic material:

$$\left. \begin{aligned} (\tilde{\lambda}\delta_{ij}\delta_{kl} + \tilde{\mu}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}))u_{k,lj} - \beta\delta_{ij}(\theta + \alpha_1\dot{\theta})_{,j} &= \rho\ddot{u}_i, \\ k\delta_{ij}\theta_{,ij} - T\beta\delta_{ij}\dot{u}_{i,j} &= \rho c(\dot{\theta} + \alpha_0\ddot{\theta}). \end{aligned} \right\} \quad (2.49)$$

Equations (2.49) may written as

$$\left. \begin{aligned} (\tilde{\lambda} + \tilde{\mu})u_{j,ij} + \tilde{\mu}u_{i,jj} - \beta(\theta + \alpha_1\dot{\theta})_{,i} &= \rho\ddot{u}_i, \\ k\theta_{,ii} - T\beta\dot{u}_{j,j} &= \rho c(\dot{\theta} + \alpha_0\ddot{\theta}). \end{aligned} \right\} \quad (2.50)$$

2.2.2 The secular equation

Now we seek solutions of (2.50) in the form of harmonic plane waves

$$\{u_i, \theta\} = \{U_i, \Theta\} \exp \{i\omega(s\mathbf{n} \cdot \mathbf{x} - t)\}, \quad (2.51)$$

where ω is the angular frequency and \mathbf{n} is the unit wave normal vector to the direction of the propagation, both of which are real constants. The amplitudes $\{U_i, \Theta\}$ and

slowness s are in general complex constants. Inserting (2.51) into (2.50) leads to the propagation conditions. Firstly, the derivatives are

$$\begin{aligned} u_{j,ij} &= -(\omega s)^2 n_i n_j U_j e^\chi, \quad (\theta + \alpha_1 \dot{\theta})_{,i} = i\omega s n_i (1 - i\omega \alpha_1) \Theta e^\chi, \quad \ddot{u}_i = -\omega^2 U_i e^\chi, \\ \theta_{,ii} &= -(\omega s)^2 n_i n_i \Theta e^\chi, \quad \dot{u}_{j,j} = \omega^2 s n_j U_j e^\chi, \quad (\dot{\theta} + \alpha_0 \ddot{\theta}) = -i\omega (1 - i\omega \alpha_0) \Theta e^\chi, \end{aligned}$$

where

$$\chi = i\omega(s\mathbf{n} \cdot \mathbf{x} - t).$$

Now substitute these derivatives into (2.50) and cancel the exponential factor e^χ to get linear algebraic equations. Firstly, from (2.50)₁ we find that

$$(\tilde{\lambda} + \tilde{\mu})(-\omega s)^2 n_i n_j U_j + \tilde{\mu}(-\omega s)^2 n_j n_i U_i - \beta(i\omega s n_i (1 - i\omega \alpha_1) \Theta) = \rho(-\omega^2) U_i.$$

Dividing this equation by $(-\omega s)^2$ we get

$$[(\tilde{\lambda} + \tilde{\mu})n_i n_j + (\tilde{\mu} - \rho s^{-2})\delta_{ij}]U_j + i\beta(\omega s)^{-1}n_i(1 - i\omega \alpha_1)\Theta = 0. \quad (2.52)$$

From (2.50)₂, after inserting the derivatives, we find that

$$k(-\omega s)^2 n_i n_i \Theta - T\beta\omega^2 s n_j U_j = \rho c(-i\omega(1 - i\omega \alpha_0))\Theta.$$

Dividing this equation by $(-\omega)$ we obtain

$$T\beta\omega s n_j U_j + (\omega s^2 k - i\rho c(1 - i\omega \alpha_0))\Theta = 0. \quad (2.53)$$

Eliminate Θ between equations (2.52) and (2.53) by writing equation (2.53) as

$$\Theta = \frac{-T\beta\omega s n_j U_j}{\omega s^2 k - i\rho c(1 - i\omega \alpha_0)}. \quad (2.54)$$

Inserting (2.54) into (2.52) we get

$$\left\{ (\tilde{\lambda} + \tilde{\mu})n_i n_j + (\tilde{\mu} - w)\delta_{ij} \right\} U_j + i\beta(\omega s)^{-1}n_i(1 - i\omega \alpha_1) \left[\frac{-T\beta\omega s n_j U_j}{\omega s^2 k - i\rho c(1 - i\omega \alpha_0)} \right] = 0, \quad (2.55)$$

where

$$w = \rho s^{-2}.$$

On rearranging equation (2.55), we obtain

$$\left\{ (\tilde{\mu} - w)\delta_{ij} + (\tilde{\lambda} + \tilde{\mu})n_i n_j + \frac{w\beta^2 T(1 - i\omega\alpha_1)n_i n_j}{\rho c(w(1 - i\omega\alpha_0) + i\omega(k/c))} \right\} U_j = 0. \quad (2.56)$$

So that there exist non-zero amplitudes U_j satisfying (2.52) and (2.53) if and only if

$$\det \left\{ (\tilde{\mu} - w)\mathbf{1} + (\tilde{\lambda} + \tilde{\mu})\mathbf{n} \otimes \mathbf{n} + \frac{w\beta^2 T(1 - i\omega\alpha_1)\mathbf{n} \otimes \mathbf{n}}{\rho c(w(1 - i\omega\alpha_0) + i\omega(k/c))} \right\} = 0. \quad (2.56a)$$

Non-dimensionalizing (2.56a), by inserting the following dimensionless quantities

$$w' = w\gamma^{-1}, \quad \tilde{\lambda}' = \tilde{\lambda}\gamma^{-1}, \quad \tilde{\mu}' = \tilde{\mu}\gamma^{-1}, \quad \omega' = \frac{\omega}{\omega^*}, \quad \varepsilon = \frac{T\beta^2}{\rho c\gamma}, \quad \alpha'_1 = \alpha_1\omega^*, \quad \alpha'_0 = \alpha_0\omega^*, \quad (2.57)$$

gives the following secular equation

$$\det \left\{ (\tilde{\lambda}'\gamma + \tilde{\mu}'\gamma)\mathbf{n} \otimes \mathbf{n} + (\tilde{\mu}'\gamma - w'\gamma)\mathbf{1} + \frac{T w' \gamma (\varepsilon \rho c \gamma T^{-1})(1 - i\omega'\alpha'_1)}{\rho c(w'\gamma(1 - i\omega'\alpha'_0) + i\omega'\omega^*(k/c))} \mathbf{n} \otimes \mathbf{n} \right\} = 0. \quad (2.58)$$

Simplifying this equation gives

$$\det \left\{ (\tilde{\mu} - w)\mathbf{1} + (\tilde{\lambda} + \tilde{\mu} + \varepsilon \frac{w(1 - i\omega\alpha_1)}{w(1 - i\omega\alpha_0) + i\omega})\mathbf{n} \otimes \mathbf{n} \right\} = 0, \quad (2.59)$$

where we have dropped the dashes for convenience. In direct notation $\mathbf{1}$ denotes the unit tensor, \mathbf{n} the wave normal vector and \otimes the dyadic product of vectors. We quote the standard identity

$$\det(\mathbf{A} + \alpha \mathbf{a} \otimes \mathbf{a}) = \det \mathbf{A} + \alpha \mathbf{a} \cdot \mathbf{A}^{\text{adj}} \mathbf{a}, \quad (2.60)$$

in which α, \mathbf{a} and \mathbf{A} are arbitrary and adj denotes the adjugate. Equation (2.60) is derived from the following property, see [30, p. 48],

$$\det(\bar{\alpha} \mathbf{A} + \bar{\beta} \mathbf{B}) = \bar{\alpha}^3 \det \mathbf{A} + \bar{\alpha}^2 \bar{\beta} \text{tr}(\mathbf{A}^{\text{adj}} \mathbf{B}) + \bar{\alpha} \bar{\beta}^2 \text{tr}(\mathbf{A} \mathbf{B}^{\text{adj}}) + \bar{\beta}^3 \det \mathbf{B}. \quad (2.61)$$

Putting $\bar{\alpha} = 1$, $\bar{\beta} = \alpha$, $\mathbf{B} = \mathbf{a} \otimes \mathbf{a}$, so that $\det \mathbf{B} = 0$ and $\mathbf{B}^{\text{adj}} = \mathbf{0}$, into this equation gives

$$\begin{aligned} \det(\mathbf{A} + \alpha \mathbf{a} \otimes \mathbf{a}) &= \det \mathbf{A} + \alpha \text{tr}(\mathbf{A}^{\text{adj}} \mathbf{a} \otimes \mathbf{a}) \\ &= \det \mathbf{A} + \alpha \mathbf{a} \cdot \mathbf{A}^{\text{adj}} \mathbf{a}. \end{aligned}$$

Applying (2.60) to (2.59), we get the secular equation in the form

$$\det \{(\tilde{\mu} - w)\mathbf{1}\} + \left(\tilde{\lambda} + \tilde{\mu} + \varepsilon \frac{w(1 - i\omega\alpha_1)}{w(1 - i\omega\alpha_0) + i\omega} \right) \mathbf{n} \cdot \{(\tilde{\mu} - w)\mathbf{1}\}^{\text{adj}} \cdot \mathbf{n} = 0. \quad (2.62)$$

In this equation we can write

$$\det \{(\tilde{\mu} - w)\mathbf{1}\} = (\tilde{\mu} - w)^3, \quad (2.63)$$

and

$$\mathbf{n} \cdot \{(\tilde{\mu} - w)\mathbf{1}\}^{\text{adj}} \mathbf{n} = (\tilde{\mu} - w)^2. \quad (2.64)$$

So we can rewrite (2.62) in the following form:

$$(\tilde{\mu} - w)^3 + \left[(\tilde{\lambda} + \tilde{\mu}) + \varepsilon \frac{w(1 - i\omega\alpha_1)}{w(1 - i\omega\alpha_0) + i\omega} \right] (\tilde{\mu} - w)^2 = 0. \quad (2.65)$$

Factorising (2.65) we get

$$(\tilde{\mu} - w)^2 \left[(\tilde{\mu} - w) + (\tilde{\lambda} + \tilde{\mu}) + \varepsilon \frac{w(1 - i\omega\alpha_1)}{w(1 - i\omega\alpha_0) + i\omega} \right] = 0. \quad (2.66)$$

After expanding and rearranging the part within square brackets of equation (2.66) we obtain

$$(\tilde{\mu} - w)^2 \{ w^2(1 - i\omega\alpha_0) - w[(1 - i\omega\alpha_0)(\tilde{\lambda} + 2\tilde{\mu}) + \varepsilon(1 - i\omega\alpha_1) - i\omega] - i\omega(\tilde{\lambda} + 2\tilde{\mu}) \} = 0. \quad (2.67)$$

This is the secular equation for unconstrained isotropic TRDTE and has not previously appeared in the literature.

The repeated root $w = \tilde{\mu}$ of (2.67) corresponds to two transverse elastic waves that are not affected by thermal effects and are neither dispersive nor attenuated. We need not discuss these any further. The remaining quadratic factor of (2.67) is

$$w^2(1 - i\omega\alpha_0) - w[(1 - i\omega\alpha_0)(\tilde{\lambda} + 2\tilde{\mu}) + \varepsilon(1 - i\omega\alpha_1) - i\omega] - i\omega(\tilde{\lambda} + 2\tilde{\mu}) = 0. \quad (2.68)$$

This equation gives two roots that correspond to two attenuating and dispersive longitudinal waves. They can be scaled by using

$$\gamma = 2\tilde{\mu} + \tilde{\lambda}.$$

But we already know that

$$\tilde{\lambda}' = \tilde{\lambda}/\gamma, \quad \tilde{\mu}' = \tilde{\mu}/\gamma.$$

So from the last two relations we find that

$$2\tilde{\mu} + \tilde{\lambda} = 1. \quad (2.69)$$

Substituting (2.69) into (2.68) we get the final form of the secular equation for unconstrained isotropic TRDTE:

$$w^2(1 - i\omega\alpha_0) - w[(1 - i\omega\alpha_0) + \varepsilon(1 - i\omega\alpha_1) - i\omega] - i\omega = 0. \quad (2.70)$$

The roots of the quadratic equation (2.70) are given by

$$w_{1,2} = \frac{1}{2(1 - i\omega\alpha_0)} \left[z_1 \pm [z_1^2 + 4i\omega(1 - i\omega\alpha_0)]^{\frac{1}{2}} \right], \quad (2.71)$$

where

$$z_1 = (1 - i\omega\alpha_0) + \varepsilon(1 - i\omega\alpha_1) - i\omega. \quad (2.72)$$

If we put $\alpha_0 = \alpha_1 = 0$, we recover the case of the classical thermoelasticity of an unconstrained isotropic material, see [29, (2.15)].

The roots (2.71) can be plotted for varying values of ε , the measure of the degree of thermoelastic coupling, as shown in Figures 2.4 and 2.5. In the uncoupled case, when $\varepsilon = 0$, the roots of (2.70) reduce to

$$w_1 = 1, \quad w_2 = \frac{-i\omega}{(1 - i\omega\alpha_0)}, \quad (2.73)$$

where w_1 represents an unattenuated, non-dispersive longitudinal wave (a purely elastic mode) and w_2 represents a diffusive mode. In all the plots of Figures 2.4 and 2.5, $\text{Im } w \leq 0$, which is the condition for linear stability. So in the unconstrained isotropic TRDTE theory case both longitudinal modes are stable.

For $\varepsilon > 0$, we investigate the nature of the modes at high and low frequencies.

Low frequency expansions

From (2.71) with small ω , we find that

$$w_1 = \frac{1}{2(1 - i\omega\alpha_0)} \left[z_1 + [z_1^2 + 4i\omega(1 - i\omega\alpha_0)]^{\frac{1}{2}} \right], \quad (2.74)$$

where

$$z_1^2 = [(1 - i\omega\alpha_0) + \varepsilon(1 - i\omega\alpha_1)]^2 - 2i\omega[(1 - i\omega\alpha_0) + \varepsilon(1 - i\omega\alpha_1)] - \omega^2.$$

After expanding (2.74) and rearranging we get

$$w_1 = \frac{1}{2(1 - i\omega\alpha_0)} \left\{ [(1 - i\omega\alpha_0) + \varepsilon(1 - i\omega\alpha_1) - i\omega] + \left[((1 - i\omega\alpha_0) + \varepsilon(1 - i\omega\alpha_1))^2 + 2i\omega((1 - i\omega\alpha_0) - \varepsilon(1 - i\omega\alpha_1)) - \omega^2 \right]^{\frac{1}{2}} \right\}.$$

Now factorising the terms within the second square brackets by

$$((1 - i\omega\alpha_0) + \varepsilon(1 - i\omega\alpha_1))^2$$

we obtain,

$$w_1 = \frac{1}{2(1 - i\omega\alpha_0)} \left\{ [(1 - i\omega\alpha_0) + \varepsilon(1 - i\omega\alpha_1) - i\omega] + [(1 - i\omega\alpha_0) + \varepsilon(1 - i\omega\alpha_1)] \left[1 + \frac{2i\omega((1 - i\omega\alpha_0) - \varepsilon(1 - i\omega\alpha_1))}{((1 - i\omega\alpha_0) + \varepsilon(1 - i\omega\alpha_1))^2} \right]^{1/2} + O(\omega^2) \right\}.$$

Using the binomial expansion we get

$$w_1 = \frac{1}{2(1 - i\omega\alpha_0)} \left\{ [(1 - i\omega\alpha_0) + \varepsilon(1 - i\omega\alpha_1) - i\omega] + [(1 - i\omega\alpha_0) + \varepsilon(1 - i\omega\alpha_1)] \left[1 + \frac{i\omega((1 - i\omega\alpha_0) - \varepsilon(1 - i\omega\alpha_1))}{((1 - i\omega\alpha_0) + \varepsilon(1 - i\omega\alpha_1))^2} \right] + O(\omega^2) \right\}.$$

Expanding the equation we obtain

$$w_1 = \frac{1}{2(1 - i\omega\alpha_0)} \left\{ 2[(1 - i\omega\alpha_0) + \varepsilon(1 - i\omega\alpha_1)] - i\omega \left[1 - \frac{(1 - i\omega\alpha_0) - \varepsilon(1 - i\omega\alpha_1)}{(1 - i\omega\alpha_0) + \varepsilon(1 - i\omega\alpha_1)} \right] \right\} + O(\omega^2).$$

Rearranging the last equation to get

$$w_1 = 1 + \varepsilon \left(\frac{1 - i\omega\alpha_1}{1 - i\omega\alpha_0} \right) - \left[\frac{i\omega\varepsilon(1 - i\omega\alpha_1)}{(1 - i\omega\alpha_0)^2 + \varepsilon(1 - i\omega\alpha_1)(1 - i\omega\alpha_0)} \right] + O(\omega^2),$$

which may be expanded further for small ω to obtain

$$w_1 = 1 + \varepsilon - i\omega\varepsilon \left\{ \alpha_1 - \alpha_0 + \frac{1}{1 + \varepsilon} \right\} + O(\omega^2). \quad (2.75)$$

Similarly, we can get w_2 to be

$$w_2 = \frac{1}{2(1 - i\omega\alpha_0)} \left[z_1 - [z_1^2 + 4i\omega(1 - i\omega\alpha_0)]^{\frac{1}{2}} \right], \quad (2.76)$$

and expanding for small ω gives

$$w_2 = \frac{-i\omega}{1 + \varepsilon} + O(\omega^2). \quad (2.77)$$

Equation (2.75) is an elastic mode and equation (2.77) a diffusive mode (marked \times on Figures 2.4 and 2.5 for $\omega = 0$).

High frequency expansions

The roots of the secular equation (2.70) in the high frequency limit, as $\omega \rightarrow \infty$, may be obtained by dividing (2.70) by $i\omega$:

$$w^2 \left(\frac{1}{i\omega} - \alpha_0 \right) - w \left[\frac{1}{i\omega} - \alpha_0 + \varepsilon \left(\frac{1}{i\omega} - \alpha_1 \right) - 1 \right] - 1 = 0.$$

Now let $i\omega \rightarrow \infty$, i.e. $\frac{1}{i\omega} \rightarrow 0$. The above secular equation becomes, for large ω ,

$$\alpha_0 w^2 - [1 + \alpha_0 + \alpha_1 \varepsilon] w + 1 = 0.$$

Define

$$H(w) \equiv \alpha_0 w^2 - [1 + \alpha_0 + \alpha_1 \varepsilon] w + 1.$$

Now we need to determine the position of the zeros of $H(w)$

$$\begin{aligned} H(0) &= 1 > 0, \\ H(1) &= -\alpha_1 \varepsilon < 0, \\ H(1 + \varepsilon) &= -\varepsilon - (\alpha_1 - \alpha_0) \varepsilon - (\alpha_1 - \alpha_0) \varepsilon^2 < 0, \\ H(\infty) &= \infty > 0. \end{aligned} \quad (2.77a)$$

So we have in the high frequency limit, real roots \bar{h}_1 and \bar{h}_2 of $H(w) = 0$ satisfying

$$0 < \bar{h}_1 < 1 < 1 + \varepsilon < \bar{h}_2. \quad (2.77b)$$

They satisfy the following quadratic polynomial

$$\bar{h}(w) = (w - \bar{h}_1)(w - \bar{h}_2),$$

and we must have

$$H(w) = \alpha_0 \bar{h}(w).$$

Collect terms in $i\omega$ in (2.70) together:

$$w^2 - w(1 + \varepsilon) - i\omega \left[\alpha_0 w^2 - (1 + \alpha_0 + \alpha_1 \varepsilon)w + 1 \right] = 0.$$

The above secular equation may be written as

$$w^2 - w(1 + \varepsilon) - i\omega \alpha_0 (w - \bar{h}_1)(w - \bar{h}_2) = 0. \quad (2.77c)$$

For ω sufficiently large the roots of (2.77c) may be written as

$$w_1 = \bar{h}_1 + \frac{A}{i\omega} \quad \text{and} \quad w_2 = \bar{h}_2 + \frac{B}{i\omega}. \quad (2.78)$$

Substituting (2.78) into (2.77c) gives

$$A = \frac{\bar{h}_1(\bar{h}_1 - (1 + \varepsilon))}{\alpha_0(\bar{h}_1 - \bar{h}_2)} > 0 \quad \text{and} \quad B = \frac{\bar{h}_2(\bar{h}_2 - (1 + \varepsilon))}{\alpha_0(\bar{h}_2 - \bar{h}_1)} > 0, \quad (2.79)$$

the fact that A and B are positive coming from the inequalities (2.77b). It follows that both branches are stable in the high frequency limit.

Both branches are stable at both low and high frequencies and so an argument similar to the one involving equation (2.46) shows that both branches are stable for all frequencies.

Numerical results

In each of Figures 2.4 and 2.5 there is a \times at zero and $1 + \varepsilon$, marking the low frequency limits and the high frequency limits are marked with a \circ .

In Figure 2.4 there are two finite branches. For low frequencies the branch beginning at the origin is diffusive for all values of ε and both branches are elastic for high frequencies. It can be seen that the two branches intersect for $\varepsilon = 0.63$, approximately.

In Figure 2.5 we see one finite branch and one infinite branch. This is because $\alpha_0 = 0$ which leads to $H(w)$ being linear in w rather than quadratic so that the second high frequency root becomes infinite. The situation is similar to that in Figure 2.2 for the anisotropic case. We see that for $\varepsilon = 0$ the left hand branch is diffusive for both low and high frequencies and the right hand branch is elastic for all frequencies; for $\varepsilon > 0$ the left hand branch is diffusive for low frequencies but elastic for high frequencies and the right hand branch is elastic for low frequencies but diffusive for high frequencies. The cross over point is at $\varepsilon = 1$ in part (d). This is described by Chadwick [7].

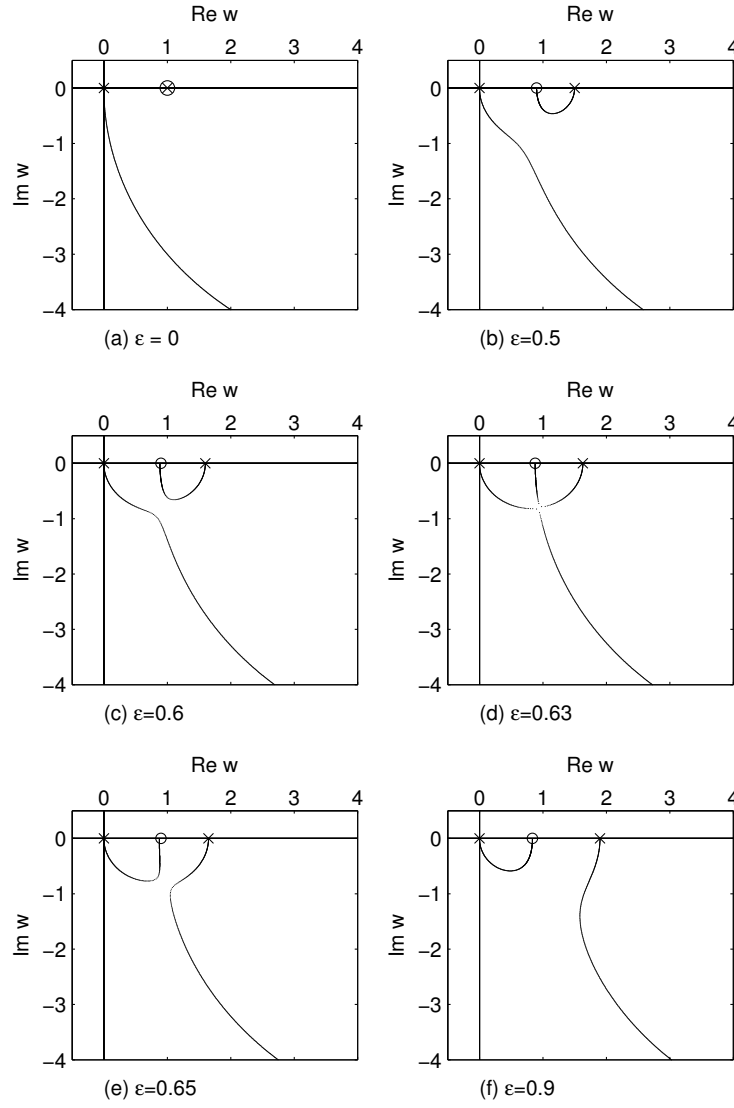


Figure 2.4: The longitudinal squared wave speeds of unconstrained isotropic TRDTE theory. For each part, $\alpha_0 = 0.1$, $\alpha_1 = 0.2$.

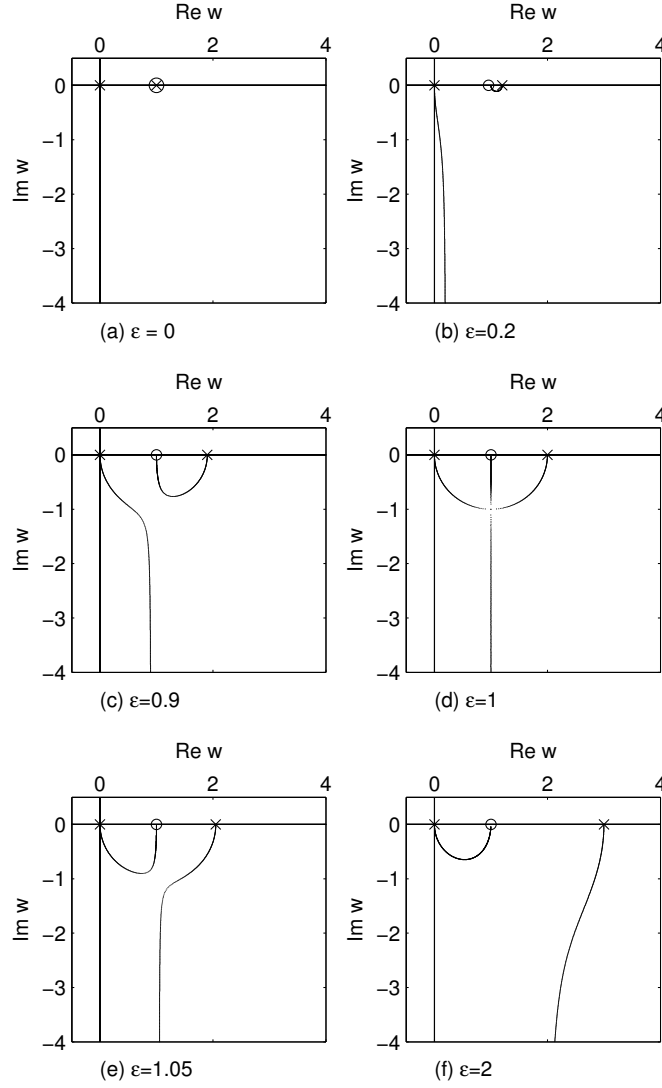


Figure 2.5: The longitudinal squared wave speeds of unconstrained isotropic classical thermoelasticity theory. For each part, $\alpha_0 = 0$, $\alpha_1 = 0$.

2.3 Constrained anisotropic TRDTE

2.3.1 Usual deformation-temperature constraints

We suppose that an elastic heat conductor B possesses an equilibrium configuration B_e and is subject to a thermomechanical constraint connecting the deformation and temperature of the form

$$f(\mathbf{F}, T + \theta) = 0, \quad \text{with} \quad f(\mathbf{F}_e, T) = 0, \quad (2.80)$$

where f is a dimensionless scalar function and \mathbf{F}_e and θ denote the deformation gradient and the temperature increment in B_e with uniform absolute temperature T , respectively. The following equation is obtained by linearising the constraint (2.80), see [16]

$$\tilde{N}_{qp} u_{p,q} - \alpha \theta = 0, \quad (2.81)$$

where $\tilde{\mathbf{N}}$ is a dimensionless symmetric constraint tensor.

Basic equations

We might assume that the stress, entropy increment and heat flux, are given by analogy with the constrained classical case; see [30, (3.4)],

$$\left. \begin{aligned} \sigma_{ij} &= \tilde{c}_{ijkl} u_{k,l} - \beta_{ij} \left(1 + \alpha_1 \frac{\partial}{\partial t} \right) \theta + \tilde{N}_{ij} \tilde{\eta}, \\ \phi &= \rho^{-1} \beta_{ij} u_{i,j} + T^{-1} c \left(1 + \alpha_0 \frac{\partial}{\partial t} \right) \theta + \rho^{-1} \alpha \tilde{\eta}, \\ q_i &= -k_{ij} \theta_{,j}. \end{aligned} \right\} \quad (2.82)$$

Most of the quantities appearing in (2.82) have been defined earlier; the new function $\tilde{\eta}(\mathbf{x}, t)$ is such that $\tilde{\eta} \tilde{\mathbf{N}}$ is a reaction stress and $\rho^{-1} \alpha \tilde{\eta}$ is a reaction entropy, see Chadwick and Scott [16]. We shall now derive the secular equation. From (2.82)₁ and (2.1)₁ we see that

$$\tilde{c}_{ijkl} u_{k,lj} - \beta_{ij} (\theta_{,j} + \alpha_1 \dot{\theta}_{,j}) + \tilde{N}_{ij} \tilde{\eta}_{,j} = \rho \ddot{u}_i. \quad (2.83)$$

From (2.82)₃ and (2.1)₂ we obtain

$$k_{ij} \theta_{,ij} = \rho T \dot{\phi}. \quad (2.84)$$

Differentiating (2.82)₂ with respect to time after multiplying by ρT , and then inserting into (2.84), gives

$$T\beta_{qp}\dot{u}_{p,q} + \rho c(\dot{\theta} + \alpha_0\ddot{\theta}) + T\alpha\dot{\tilde{\eta}} - k_{ij}\theta_{,ij} = 0. \quad (2.85)$$

The secular equation

Now we seek to find the solutions of (2.81), (2.83) and (2.85) in the form of plane harmonic waves

$$\{u_i, \theta, \tilde{\eta}\} = \{U_i, \Theta, \tilde{H}\} \exp \{i\omega(s\mathbf{n} \cdot \mathbf{x} - t)\}, \quad (2.86)$$

similarly to (2.4) in the unconstrained case. We insert (2.86) into (2.83), (2.84) and (2.85). We found all the derivatives $u_{k,lj}, \theta_{,j}, \ddot{u}_i, \theta_{,ij}$ and $\dot{u}_{i,j}$ previously and now just observe that

$$\begin{aligned} \dot{\theta}_{,j} &= \omega^2 s n_j \Theta e^\chi, \quad \tilde{\eta}_{,j} = (i\omega s n_j) \tilde{H} e^\chi, \quad \dot{\theta} = -i\omega \Theta e^\chi, \\ \ddot{\theta} &= -\omega^2 \Theta e^\chi, \quad \dot{\tilde{\eta}} = -i\omega \tilde{H} e^\chi, \quad u_{p,q} = i\omega s n_q U_p e^\chi, \end{aligned}$$

where the phase factor is given once more by

$$\chi = i\omega(s\mathbf{n} \cdot \mathbf{x} - t).$$

Substituting all these derivatives into (2.83), (2.84) and (2.85). Firstly, (2.83) becomes

$$\tilde{c}_{ijkl}(-\omega^2 s^2 n_l n_j) U_k - \beta_{ij}(i\omega s n_j + \alpha_1 \omega^2 s n_j) \Theta + \tilde{N}_{ij}(i\omega s n_j) \tilde{H} = \rho(-\omega^2) U_i. \quad (2.87)$$

Dividing all terms by $(-\omega^2 s^2)$ we get

$$\tilde{c}_{ijkl} n_l n_j U_k + \beta_{ij} n_j i(\omega s)^{-1} (1 - i\omega \alpha_1) \Theta - \tilde{N}_{ij} n_j i(\omega s)^{-1} \tilde{H} = \rho s^{-2} U_i. \quad (2.88)$$

This equation can be rewritten as

$$(\tilde{Q}_{ik} - \rho s^{-2} \delta_{ik}) U_k + i(\omega s)^{-1} [b_i (1 - i\omega \alpha_1) \Theta - \tilde{c}_i \tilde{H}] = 0, \quad (2.89)$$

in which we define $\tilde{Q}_{ik} = \tilde{c}_{ijkl} n_j n_l$, $b_i = \beta_{ij} n_j$, $\tilde{c}_i = \tilde{N}_{ij} n_j$. This equation is different from Chadwick and Scott [16, (4.2)₁] because of the presence of α_1 . When $\alpha_1 = 0$ this equation reverts to [16, (4.2)₁], the equation of the constrained anisotropic classical case. Equation (2.85) becomes

$$T\beta_{qp}(\omega^2 s n_q) U_p + \rho c(-i\omega + \alpha_0(-\omega^2)) \Theta + T\alpha(-i\omega \tilde{H}) - k_{ij}(-\omega^2 s^2 n_i n_j) \Theta = 0. \quad (2.90)$$

After dividing by ω and rearranging the equation we get

$$\omega s T b_p U_p - i \alpha T \tilde{H} + (\omega s^2 k - i \rho c (1 - i \omega \alpha_0)) \Theta = 0, \quad (2.91)$$

where we define $k = k_{ij} n_i n_j$. The above equation reduces to [16, (4.2)₂] when $\alpha_0 = 0$, the classical case. Finally, equation (2.81) becomes

$$i \omega s \tilde{c}_p U_p - \alpha \Theta = 0, \quad (2.92)$$

which is similar to its classical counterpart [16, (4.2)₃]. We must eliminate Θ and \tilde{H} between equations (2.89), (2.91) and (2.92). So, we need to rewrite (2.92) as

$$\Theta = \alpha^{-1} i \omega s \tilde{c}_p U_p. \quad (2.93)$$

Substituting (2.93) into (2.91) gives

$$\tilde{H} = -i \alpha^{-1} \omega s b_p U_p + (\alpha^2 T)^{-1} \omega s \tilde{c}_p U_p (\omega s^2 k - i \rho c (1 - i \omega \alpha_0)). \quad (2.94)$$

Inserting (2.93) and (2.94) into (2.89), then rearranging the equation, gives

$$\begin{aligned} \{ \tilde{Q}_{ip} - \alpha^{-1} (b_i \tilde{c}_p (1 - i \omega \alpha_1) + \tilde{c}_i b_p) - (\alpha^2 T)^{-1} (\rho c (1 - i \omega \alpha_0) + i \omega s^2 k) \tilde{c}_i \tilde{c}_p \\ - \rho s^{-2} \delta_{ip} \} U_p = 0. \end{aligned} \quad (2.95)$$

Rearranging this equation we get

$$\begin{aligned} \{ \tilde{Q}_{ip} - \alpha^{-1} (b_i \tilde{c}_p + \tilde{c}_i b_p) - (\alpha^2 T)^{-1} (\rho c + i \omega s^2 k) \tilde{c}_i \tilde{c}_p + (\alpha^2 T)^{-1} \rho c i \omega \alpha_0 \tilde{c}_i \tilde{c}_p + \alpha^{-1} i \omega \alpha_1 b_i \tilde{c}_p \\ - \rho s^{-2} \delta_{ip} \} U_p = 0. \end{aligned}$$

The non-zero amplitudes satisfy (2.89), (2.91) and (2.92) if and only if

$$\begin{aligned} \det \{ \tilde{\mathbf{Q}} - \alpha^{-1} (\mathbf{b} \otimes \tilde{\mathbf{c}} + \tilde{\mathbf{c}} \otimes \mathbf{b}) - (\alpha^2 T)^{-1} (\rho c + i \omega s^2 k) \tilde{\mathbf{c}} \otimes \tilde{\mathbf{c}} + (\alpha^2 T)^{-1} \rho c i \omega \alpha_0 \tilde{\mathbf{c}} \otimes \tilde{\mathbf{c}} \\ + \alpha^{-1} i \omega \alpha_1 \mathbf{b} \otimes \tilde{\mathbf{c}} - \rho s^{-2} \mathbf{1} \} = 0. \end{aligned} \quad (2.96)$$

By defining

$$\tilde{\mathbf{P}} := \tilde{\mathbf{Q}} - \alpha^{-1} (\mathbf{b} \otimes \tilde{\mathbf{c}} + \tilde{\mathbf{c}} \otimes \mathbf{b}) - \frac{\rho c}{\alpha^2 T} \tilde{\mathbf{c}} \otimes \tilde{\mathbf{c}}, \quad (2.97)$$

we may rewrite (2.96) as

$$\det \{ (\tilde{\mathbf{P}} - w \mathbf{1}) + \frac{i \omega \alpha_1}{\alpha} \mathbf{b} \otimes \tilde{\mathbf{c}} - \frac{i \omega}{\alpha^2 T} (s^2 k - \rho c \alpha_0) \tilde{\mathbf{c}} \otimes \tilde{\mathbf{c}} \} = 0, \quad (2.98)$$

where

$$w = \rho s^{-2}.$$

Equation (2.98) may be rewritten as

$$\det \left\{ (\tilde{\mathbf{P}} - w\mathbf{1}) + \left[\frac{i\omega\alpha_1}{\alpha} \mathbf{b} - \frac{i\omega}{\alpha^2 T} (s^2 k - \rho c \alpha_0) \tilde{\mathbf{c}} \right] \otimes \tilde{\mathbf{c}} \right\} = 0. \quad (2.99)$$

Using the standard identity (2.60)

$$\det\{\mathbf{A} + \alpha \bar{\mathbf{a}} \otimes \mathbf{a}\} = \det \mathbf{A} + \alpha \bar{\mathbf{a}} \cdot \mathbf{A}^{\text{adj}} \mathbf{a}, \quad (2.100)$$

and taking

$$\bar{\mathbf{a}} \equiv \frac{i\omega\alpha_1}{\alpha} \mathbf{b} - \frac{i\omega}{\alpha^2 T} (s^2 k - \rho c \alpha_0) \tilde{\mathbf{c}} \quad \text{and} \quad \mathbf{a} \equiv \tilde{\mathbf{c}},$$

equation (2.99) may be rewritten as

$$\det(\tilde{\mathbf{P}} - w\mathbf{1}) + \left[\frac{i\omega\alpha_1}{\alpha} \mathbf{b} - \frac{i\omega}{\alpha^2 T} \left(\frac{\rho k}{w} - \rho c \alpha_0 \right) \tilde{\mathbf{c}} \right] \cdot (\tilde{\mathbf{P}} - w\mathbf{1})^{\text{adj}} \tilde{\mathbf{c}} = 0. \quad (2.101)$$

This is the secular equation for anisotropic TRDTE which is constrained by the usual deformation temperature constraint and has not previously appeared in the literature.

We can rewrite (2.97) in terms of the isentropic acoustic tensor by using the definition (2.13)

$$\tilde{\mathbf{P}} := \hat{\mathbf{Q}} - \frac{T}{\rho c} \left(\mathbf{b} + \frac{\rho c}{\alpha T} \tilde{\mathbf{c}} \right) \otimes \left(\mathbf{b} + \frac{\rho c}{\alpha T} \tilde{\mathbf{c}} \right). \quad (2.102)$$

To rewrite (2.101) in a more clear form, we non-dimensionalize by introducing γ , which has the dimensions of stress, and the frequency $\omega^* = \gamma c/k$. We define the non-dimensional quantities, see [27, (2.12)],

$$\begin{aligned} w' = \gamma^{-1} w, \quad \omega' = \frac{\omega}{\omega^*}, \quad \tilde{\mathbf{P}}' = \gamma^{-1} \tilde{\mathbf{P}}, \quad \alpha' = \alpha T, \quad c' = \frac{\rho c T}{\gamma}, \quad \tilde{\mathbf{c}}' = (\tilde{\mathbf{c}} \cdot \tilde{\mathbf{c}})^{-\frac{1}{2}} \tilde{\mathbf{c}}, \quad \beta^2 = \mathbf{b} \cdot \mathbf{b}, \\ \alpha'_0 = \alpha_0 \omega^*, \quad \alpha'_1 = \alpha_1 \omega^*, \quad \varepsilon = \frac{T(\mathbf{b} \cdot \mathbf{b})}{\rho c \gamma}, \quad \mathbf{b}' = (\mathbf{b} \cdot \mathbf{b})^{-\frac{1}{2}} \mathbf{b}, \quad \tilde{\mathbf{Q}}' = \gamma^{-1} \tilde{\mathbf{Q}}, \quad \hat{\mathbf{Q}}' = \gamma^{-1} \hat{\mathbf{Q}}. \end{aligned} \quad (2.103)$$

In terms of these non-dimensionalizations we can rewrite (2.13) as, see [26, (2.14)],

$$\hat{\mathbf{Q}}' = \tilde{\mathbf{Q}}' + \varepsilon \mathbf{b}' \otimes \mathbf{b}'. \quad (2.103a)$$

The dimensionless form of (2.102) is

$$\tilde{\mathbf{P}}' := \hat{\mathbf{Q}}' - \left(\varepsilon^{\frac{1}{2}} \mathbf{b}' + \tilde{\sigma} \tilde{\mathbf{c}}' \right) \otimes \left(\varepsilon^{\frac{1}{2}} \mathbf{b}' + \tilde{\sigma} \tilde{\mathbf{c}}' \right), \quad (2.104)$$

in which for $\alpha \neq 0$ we have defined

$$\tilde{\sigma} := \frac{c'^{\frac{1}{2}} (\tilde{\mathbf{c}} \cdot \tilde{\mathbf{c}})^{\frac{1}{2}}}{\alpha'}. \quad (2.105)$$

This is the ratio of the relative importance of the mechanical and thermal parts of the constraint because $\tilde{\sigma} \rightarrow 0$ provides a purely thermal constraint ($\tilde{\mathbf{c}} \rightarrow \mathbf{0}$) and $\tilde{\sigma} \rightarrow \infty$ provides a purely mechanical constraint ($\alpha' \rightarrow 0$). Now the secular equation (2.101) can be written in terms of non-dimensional quantities as

$$\det(w' \mathbf{1} - \tilde{\mathbf{P}}') - i\omega' [\alpha'_1 \varepsilon^{\frac{1}{2}} \tilde{\sigma} \mathbf{b}' - (1/w' - \alpha'_0) \tilde{\sigma}^2 \tilde{\mathbf{c}}'] \cdot (w' \mathbf{1} - \tilde{\mathbf{P}}')^{\text{adj}} \tilde{\mathbf{c}}' = 0. \quad (2.106)$$

Rearranging this equation and dropping the dashes for convenience

$$w \det(w \mathbf{1} - \tilde{\mathbf{P}}) - i\omega \tilde{\sigma} [w \alpha_1 \varepsilon^{\frac{1}{2}} \mathbf{b} - (1 - \alpha_0 w) \tilde{\sigma} \tilde{\mathbf{c}}] \cdot (w \mathbf{1} - \tilde{\mathbf{P}})^{\text{adj}} \tilde{\mathbf{c}} = 0. \quad (2.107)$$

This equation is quartic in w .

Stability

$\tilde{\mathbf{P}}$ is a symmetric tensor and so has real eigenvalues which can be ordered so as

$$\tilde{p}_1 \leq \tilde{p}_2 \leq \tilde{p}_3. \quad (2.107a)$$

In terms of $\hat{\mathbf{Q}}$ with aid of the definition (2.104) for $\tilde{\mathbf{P}}$ we get the inequalities, see [27, (2.19)],

$$\tilde{p}_1 < \hat{q}_1 < \tilde{p}_2 < \hat{q}_2 < \tilde{p}_3 < \hat{q}_3, \quad (2.107b)$$

where \hat{q}_i , $i = 1, 2, 3$, denote the eigenvalues of $\hat{\mathbf{Q}}$ defined by (2.13). With \tilde{q}_i denoting the eigenvalues of $\tilde{\mathbf{Q}}$, the interlacing property

$$\tilde{q}_1 < \hat{q}_1 < \tilde{q}_2 < \hat{q}_2 < \tilde{q}_3 < \hat{q}_3 \quad (2.107c)$$

was proved in [9]. From (2.107a) and (2.107b) we can deduce that the signs of \tilde{p}_2 and \tilde{p}_3 are positive because $\hat{q}_1 > 0$ but \tilde{p}_1 could be positive or negative. Equation (2.107) may be rewritten as

$$w \det(w \mathbf{1} - \tilde{\mathbf{P}}) - i\omega \tilde{\sigma} \left\{ w \alpha_1 \varepsilon^{\frac{1}{2}} [\mathbf{b} \cdot (w \mathbf{1} - \tilde{\mathbf{P}})^{\text{adj}} \tilde{\mathbf{c}}] - \tilde{\sigma} (1 - \alpha_0 w) [\tilde{\mathbf{c}} \cdot (w \mathbf{1} - \tilde{\mathbf{P}})^{\text{adj}} \tilde{\mathbf{c}}] \right\} = 0. \quad (2.108)$$

In equation (2.108) we need to group the terms in w together:

$$w \det(w\mathbf{1} - \tilde{\mathbf{P}}) - i\omega\tilde{\sigma} \left\{ w [\alpha_1 \varepsilon^{\frac{1}{2}} \mathbf{b} + \alpha_0 \tilde{\sigma} \tilde{\mathbf{c}}] \cdot (w\mathbf{1} - \tilde{\mathbf{P}})^{\text{adj}} \tilde{\mathbf{c}} - \tilde{\sigma} \tilde{\mathbf{c}} \cdot (w\mathbf{1} - \tilde{\mathbf{P}})^{\text{adj}} \tilde{\mathbf{c}} \right\} = 0. \quad (2.109)$$

Now from (2.109) we see that the zeros of

$$\tilde{\mathbf{c}} \cdot (w\mathbf{1} - \tilde{\mathbf{P}})^{\text{adj}} \tilde{\mathbf{c}} \quad (2.109a)$$

are important. This expression is quadratic in w and has zeros $w = \tilde{W}_1, \tilde{W}_2$, satisfying the following inequalities, see [27, (2.23)],

$$\tilde{p}_1 < \tilde{W}_1 < \tilde{p}_2 < \tilde{W}_2 < \tilde{p}_3. \quad (2.109b)$$

But consider the following expression in (2.109)

$$[\alpha_1 \varepsilon^{\frac{1}{2}} \mathbf{b} + \alpha_0 \tilde{\sigma} \tilde{\mathbf{c}}] \cdot (w\mathbf{1} - \tilde{\mathbf{P}})^{\text{adj}} \tilde{\mathbf{c}}.$$

In the coordinate system based on $\tilde{\mathbf{P}}$, in which

$$\tilde{\mathbf{P}} = \begin{pmatrix} \tilde{p}_1 & 0 & 0 \\ 0 & \tilde{p}_2 & 0 \\ 0 & 0 & \tilde{p}_3 \end{pmatrix},$$

we have

$$J(w) \equiv [\alpha_1 \varepsilon^{\frac{1}{2}} \mathbf{b} + \alpha_0 \tilde{\sigma} \tilde{\mathbf{c}}] \cdot (w\mathbf{1} - \tilde{\mathbf{P}})^{\text{adj}} \tilde{\mathbf{c}} = [\alpha_1 \varepsilon^{\frac{1}{2}} b_1 + \alpha_0 \tilde{\sigma} \tilde{c}_1] \tilde{c}_1 (w - \tilde{p}_2)(w - \tilde{p}_3) + \\ [\alpha_1 \varepsilon^{\frac{1}{2}} b_2 + \alpha_0 \tilde{\sigma} \tilde{c}_2] \tilde{c}_2 (w - \tilde{p}_1)(w - \tilde{p}_3) + [\alpha_1 \varepsilon^{\frac{1}{2}} b_3 + \alpha_0 \tilde{\sigma} \tilde{c}_3] \tilde{c}_3 (w - \tilde{p}_1)(w - \tilde{p}_2).$$

Now, in order to examine the sign changes of $J(w)$, we must look at, for example, the sign of

$$J(\tilde{p}_1) = (\alpha_1 \varepsilon^{\frac{1}{2}} b_1 + \alpha_0 \tilde{\sigma} \tilde{c}_1) \tilde{c}_1 (\tilde{p}_1 - \tilde{p}_2)(\tilde{p}_1 - \tilde{p}_3).$$

We do not know the signs of b_1 and \tilde{c}_1 ; they might have opposite signs. Due to the occurrence of \mathbf{b} with \mathbf{c} in $J(w)$, the sign changes would be difficult to determine. So we need to deal with a special case and choose the constraint of incompressibility at uniform temperature because for this constraint we shall see that both \mathbf{b} and \mathbf{c}

are parallel to \mathbf{n} . For incompressibility at uniform temperature the deformation-temperature constraint is, see [16, (3.3)],

$$u_{j,j} - \alpha\theta = 0. \quad (2.110)$$

In this special case $\tilde{\mathbf{N}}$ is defined as

$$\tilde{N}_{ij} = \delta_{ij}.$$

So $\tilde{\mathbf{c}}$ might be written as

$$\tilde{c}_i = \delta_{ij}n_j = n_i$$

and so

$$\tilde{\mathbf{c}} = \mathbf{In} = \mathbf{n}. \quad (2.110a)$$

Consider the special case when β is isotropic (even if the material is otherwise anisotropic), so that

$$\beta_{ij} = \beta\delta_{ij},$$

which we shall call thermal isotropy. Then

$$\mathbf{b} = \beta\mathbf{n} = \beta\mathbf{In} = \beta\mathbf{n} \quad (2.110b)$$

and in component notation \mathbf{b} might be written as

$$b_i = \beta n_i.$$

From the dimensionless quantities (2.103) we find that

$$\mathbf{b}' = (\mathbf{b} \cdot \mathbf{b})^{-\frac{1}{2}}\mathbf{b},$$

so that

$$\mathbf{b} \cdot \mathbf{b} = b_i \cdot b_i = \beta n_i \cdot \beta n_i = \beta^2.$$

Then

$$\mathbf{b}' = (\beta^2)^{-\frac{1}{2}}\mathbf{b} = \beta^{-1}\beta\mathbf{n} = \mathbf{n}.$$

Dropping dashes for convenience leads to

$$\mathbf{b} = \mathbf{n}.$$

Then

$$\begin{aligned}
J(w) &= [\alpha_1 \varepsilon^{\frac{1}{2}} \mathbf{b} + \alpha_0 \tilde{\sigma} \tilde{\mathbf{c}}] \cdot (w\mathbf{1} - \tilde{\mathbf{P}})^{\text{adj}} \tilde{\mathbf{c}}, \\
&= [\alpha_1 \varepsilon^{\frac{1}{2}} \mathbf{n} + \alpha_0 \tilde{\sigma} \mathbf{n}] \cdot (w\mathbf{1} - \tilde{\mathbf{P}})^{\text{adj}} \mathbf{n}, \\
&= [\alpha_1 \varepsilon^{\frac{1}{2}} + \alpha_0 \tilde{\sigma}] \mathbf{n} \cdot (w\mathbf{1} - \tilde{\mathbf{P}})^{\text{adj}} \mathbf{n}.
\end{aligned}$$

Thus, (2.103a) may be written in a more simplified form as

$$\hat{\mathbf{Q}} = \tilde{\mathbf{Q}} + \varepsilon \mathbf{n} \otimes \mathbf{n}, \quad (2.110c)$$

where we have dropped the dashes for convenience. This will simplify $\tilde{\mathbf{P}}$ to a more clear form; from equation (2.97)

$$\begin{aligned}
\tilde{\mathbf{P}} &= \tilde{\mathbf{Q}} - \alpha^{-1}(\beta \mathbf{n} \otimes \mathbf{n} + \beta \mathbf{n} \otimes \mathbf{n}) - \frac{\rho c}{\alpha^2 T} \mathbf{n} \otimes \mathbf{n}, \\
\Rightarrow \quad \tilde{\mathbf{P}} &= \tilde{\mathbf{Q}} - (2\alpha^{-1}\beta + \frac{\rho c}{\alpha^2 T}) \mathbf{n} \otimes \mathbf{n}.
\end{aligned} \quad (2.111)$$

The dimensionless form of (2.111) may be written as

$$\tilde{\mathbf{P}}' = \tilde{\mathbf{Q}}' - \tilde{\sigma}(2\varepsilon^{\frac{1}{2}} + \tilde{\sigma}) \mathbf{n} \otimes \mathbf{n} \quad \text{or} \quad \tilde{\mathbf{P}} = \tilde{\mathbf{Q}} - \{(\varepsilon^{\frac{1}{2}} + \tilde{\sigma})^2 - \varepsilon\} \mathbf{n} \otimes \mathbf{n}, \quad (2.112)$$

dropping the dashes, where $\tilde{\sigma}$ here is defined by

$$\tilde{\sigma} = \frac{c'^{\frac{1}{2}}(\tilde{\mathbf{n}} \cdot \tilde{\mathbf{n}})^{\frac{1}{2}}}{\alpha'} = \frac{c'^{\frac{1}{2}}}{\alpha'}.$$

Now the secular equation (2.109) might be written in non-dimensionlised form as

$$w \det(w\mathbf{1} - \tilde{\mathbf{P}}) - i\omega \tilde{\sigma} \left\{ w(\alpha_1 \varepsilon^{\frac{1}{2}} + \alpha_0 \tilde{\sigma}) \mathbf{n} \cdot (w\mathbf{1} - \tilde{\mathbf{P}})^{\text{adj}} \mathbf{n} - \tilde{\sigma} \mathbf{n} \cdot (w\mathbf{1} - \tilde{\mathbf{P}})^{\text{adj}} \mathbf{n} \right\} = 0.$$

Rearranging this equation gives another version of the secular equation:

$$w \det(w\mathbf{1} - \tilde{\mathbf{P}}) - i\omega \tilde{\sigma} [w(\alpha_1 \varepsilon^{\frac{1}{2}} + \alpha_0 \tilde{\sigma}) - \tilde{\sigma}] \mathbf{n} \cdot (w\mathbf{1} - \tilde{\mathbf{P}})^{\text{adj}} \mathbf{n} = 0. \quad (2.113)$$

$\tilde{\mathbf{P}}^{\text{adj}}$ has eigenvalues $\tilde{p}_2 \tilde{p}_3, \tilde{p}_3 \tilde{p}_1, \tilde{p}_1 \tilde{p}_2$, enabling us to rewrite the secular equation (2.113) in terms of $\tilde{p}_1, \tilde{p}_2, \tilde{p}_3$ as

$$\begin{aligned}
&w(w - \tilde{p}_1)(w - \tilde{p}_2)(w - \tilde{p}_3) - i\omega \tilde{\sigma} [w(\alpha_1 \varepsilon^{\frac{1}{2}} + \alpha_0 \tilde{\sigma}) - \tilde{\sigma}] \\
&\quad \{n_1^2(w - \tilde{p}_2)(w - \tilde{p}_3) + n_2^2(w - \tilde{p}_1)(w - \tilde{p}_3) + n_3^2(w - \tilde{p}_1)(w - \tilde{p}_2)\} = 0.
\end{aligned} \quad (2.114)$$

The quadratic part within braces has the same zeros as the expression (2.109a), namely, \tilde{W}_i , $i = 1, 2$. Let us define the quadratic part to be

$$h(w) \equiv n_1^2(w - \tilde{p}_2)(w - \tilde{p}_3) + n_2^2(w - \tilde{p}_1)(w - \tilde{p}_3) + n_3^2(w - \tilde{p}_1)(w - \tilde{p}_2).$$

Now we need to examine the sign changes of $h(w)$:

$$h(0) = n_1^2\tilde{p}_2\tilde{p}_3 + n_2^2\tilde{p}_1\tilde{p}_3 + n_3^2\tilde{p}_1\tilde{p}_2 > 0,$$

$$h(\tilde{p}_1) = n_1^2(\tilde{p}_1 - \tilde{p}_2)(\tilde{p}_1 - \tilde{p}_3) > 0,$$

$$h(\tilde{p}_2) = n_2^2(\tilde{p}_2 - \tilde{p}_3)(\tilde{p}_2 - \tilde{p}_1) < 0,$$

$$h(\tilde{p}_3) = n_3^2(\tilde{p}_3 - \tilde{p}_1)(\tilde{p}_3 - \tilde{p}_2) > 0,$$

$$h(\infty) = \infty > 0.$$

From these inequalities we may determine the positions of the zeros of $h(w)$, so that \tilde{W}_1 is between \tilde{p}_1 and \tilde{p}_2 and \tilde{W}_2 is between \tilde{p}_2 and \tilde{p}_3 . Therefore, we get the same inequalities as (2.109b):

$$\tilde{p}_1 < \tilde{W}_1 < \tilde{p}_2 < \tilde{W}_2 < \tilde{p}_3.$$

Then the secular equation (2.114) can be rewritten as

$$\tilde{F}(w) - i\omega\tilde{\sigma}[w(\alpha_1\varepsilon^{\frac{1}{2}} + \alpha_0\tilde{\sigma}) - \tilde{\sigma}]\tilde{G}(w) = 0, \quad (2.115)$$

where

$$\tilde{F}(w) = w \prod_{i=1}^3 (w - \tilde{p}_i), \quad \tilde{G}(w) = \prod_{i=1}^2 (w - \tilde{W}_i). \quad (2.116)$$

Putting $\alpha_1 = \alpha_0 = 0$ in the secular equation (2.115) we will get the secular equation of anisotropic material that is constrained by deformation temperature constraint in classical thermoelasticity, see [30, (3.6)].

Low frequency expansions

When $\omega = 0$, the roots of the secular equation (2.115) $w_i, i = 0, 1, 2, 3$, are the zeros of $\tilde{F}(w) : w = \tilde{p}_i, i = 0, 1, 2, 3$, defining $\tilde{p}_0 = 0$. Taylor expansions of the roots of the secular equation (2.115) take the form

$$w_i(\omega) = \tilde{p}_i + \sum_{n=1}^{\infty} d_n^{(i)} (-i\omega)^n, \quad i = 0, 1, 2, 3. \quad (2.117)$$

We can get the branches $w_i, i = 0, 1, 2, 3$, by inserting (2.117) into (2.115). Firstly, when $i = 0, n = 1$

$$w_0(\omega) = \tilde{p}_0 + d_1^{(0)}(-i\omega) + O(\omega^2). \quad (2.118)$$

Substituting (2.118) into (2.115) we get

$$d_1^{(0)} = -\tilde{\sigma}^2 \frac{\tilde{W}_1 \tilde{W}_2}{\tilde{p}_1 \tilde{p}_2 \tilde{p}_3}. \quad (2.119)$$

The sign of $d_1^{(0)}$ depends on the sign of \tilde{p}_1 . Now by inserting (2.119) into (2.118) we get

$$w_0(\omega) = i\omega \tilde{\sigma} \frac{\tilde{G}(0)}{\tilde{F}(0)} + O(\omega^2). \quad (2.120)$$

It is clear that from (2.119) that if $\tilde{p}_1 > 0$ then $d_1^{(0)} < 0$, thus $\text{Im } w_0(\omega) > 0$, and so $w_0(\omega)$ is unstable. But if $\tilde{p}_1 < 0$ then $d_1^{(0)} > 0$ and $w_0(\omega)$ is stable.

The exceptional case $\tilde{p}_1 = 0$ will be dealt with later. It represents a cross over for the branch $w_0(\omega)$ between instability for $\tilde{p}_1 > 0$ and stability for $\tilde{p}_1 < 0$.

When $i = 1, n = 1$, we get

$$w_1(\omega) = \tilde{p}_1 + d_1^{(1)}(-i\omega) + O(\omega^2). \quad (2.121)$$

By inserting (2.121) into the secular equation (2.115) we get

$$d_1^{(1)} = -\tilde{\sigma} [\tilde{p}_1(\alpha_1 \varepsilon^{\frac{1}{2}} + \alpha_0 \tilde{\sigma}) - \tilde{\sigma}] \frac{(\tilde{p}_1 - \tilde{W}_1)(\tilde{p}_1 - \tilde{W}_2)}{\tilde{p}_1(\tilde{p}_1 - \tilde{p}_2)(\tilde{p}_1 - \tilde{p}_3)}. \quad (2.122)$$

The sign of $d_1^{(1)}$ depends on the signs of $[\tilde{p}_1(\alpha_1 \varepsilon^{\frac{1}{2}} + \alpha_0 \tilde{\sigma}) - \tilde{\sigma}]$ and \tilde{p}_1 .

So, if $\tilde{p}_1 < 0$ then $d_1^{(1)} < 0$, thus $w_1(\omega)$ is unstable.

Stability is satisfied if $d_1^{(1)} > 0$, and $d_1^{(1)}$ is positive if

$$0 < \tilde{p}_1 < \frac{\tilde{\sigma}}{(\alpha_1 \varepsilon^{\frac{1}{2}} + \alpha_0 \tilde{\sigma})}.$$

Substituting (2.122) into (2.121) we get, for all values of \tilde{p}_1 ,

$$w_1(\omega) = \tilde{p}_1 + i\omega \tilde{\sigma} [\tilde{p}_1(\alpha_1 \varepsilon^{\frac{1}{2}} + \alpha_0 \tilde{\sigma}) - \tilde{\sigma}] \frac{\tilde{G}(\tilde{p}_1)}{\tilde{F}'(\tilde{p}_1)} + O(\omega^2). \quad (2.123)$$

The exceptional case $\tilde{p}_1 = 0$ is considered later.

Similarly, when $i = 2, 3$ and $n = 1$, we find that

$$w_2(\omega) = \tilde{p}_2 + d_1^{(2)}(-i\omega) + O(\omega^2). \quad (2.123a)$$

$$w_3(\omega) = \tilde{p}_3 + d_1^{(3)}(-i\omega) + O(\omega^2). \quad (2.123b)$$

By substituting (2.123a) and (2.123b) into (2.115) we get, respectively,

$$d_1^{(2)} = -\tilde{\sigma} [\tilde{p}_2(\alpha_1 \varepsilon^{\frac{1}{2}} + \alpha_0 \tilde{\sigma}) - \tilde{\sigma}] \frac{(\tilde{p}_2 - \tilde{W}_1)(\tilde{p}_2 - \tilde{W}_2)}{\tilde{p}_2(\tilde{p}_2 - \tilde{p}_1)(\tilde{p}_2 - \tilde{p}_3)}, \quad (2.124)$$

$$d_1^{(3)} = -\tilde{\sigma} [\tilde{p}_3(\alpha_1 \varepsilon^{\frac{1}{2}} + \alpha_0 \tilde{\sigma}) - \tilde{\sigma}] \frac{(\tilde{p}_3 - \tilde{W}_1)(\tilde{p}_3 - \tilde{W}_2)}{\tilde{p}_3(\tilde{p}_3 - \tilde{p}_1)(\tilde{p}_3 - \tilde{p}_2)}. \quad (2.125)$$

Again, stability is satisfied when $d_1^{(2)}$ and $d_1^{(3)}$ are positive, so that means when $[\tilde{p}_2(\alpha_1 \varepsilon^{\frac{1}{2}} + \alpha_0 \tilde{\sigma}) - \tilde{\sigma}]$ and $[\tilde{p}_3(\alpha_1 \varepsilon^{\frac{1}{2}} + \alpha_0 \tilde{\sigma}) - \tilde{\sigma}]$ are negative, respectively. It is clear that these signs are unaffected by \tilde{p}_1 being positive, negative or zero. Substituting (2.124) and (2.125) into (2.123a) and (2.123b), respectively, we get

$$w_2(\omega) = \tilde{p}_2 + i\omega \tilde{\sigma} [\tilde{p}_2(\alpha_1 \varepsilon^{\frac{1}{2}} + \alpha_0 \tilde{\sigma}) - \tilde{\sigma}] \frac{\tilde{G}(\tilde{p}_2)}{\tilde{F}'(\tilde{p}_2)} + O(\omega^2).$$

$$w_3(\omega) = \tilde{p}_3 + i\omega \tilde{\sigma} [\tilde{p}_3(\alpha_1 \varepsilon^{\frac{1}{2}} + \alpha_0 \tilde{\sigma}) - \tilde{\sigma}] \frac{\tilde{G}(\tilde{p}_3)}{\tilde{F}'(\tilde{p}_3)} + O(\omega^2).$$

If $[\tilde{p}_1(\alpha_1 \varepsilon^{\frac{1}{2}} + \alpha_0 \tilde{\sigma}) - \tilde{\sigma}] > 0$ then it follows that $[\tilde{p}_i(\alpha_1 \varepsilon^{\frac{1}{2}} + \alpha_0 \tilde{\sigma}) - \tilde{\sigma}] > 0$, for $i = 2, 3$, and so the instability of w_1 forces the instability of w_2 and w_3 . Conversely, if $[\tilde{p}_3(\alpha_1 \varepsilon^{\frac{1}{2}} + \alpha_0 \tilde{\sigma}) - \tilde{\sigma}] < 0$ then it follows that $[\tilde{p}_i(\alpha_1 \varepsilon^{\frac{1}{2}} + \alpha_0 \tilde{\sigma}) - \tilde{\sigma}] < 0$, for $i = 1, 2$, and so the stability of w_3 forces the stability of w_1 and w_2 . This is illustrated in Figures 2.6 and 2.7.

High frequency expansions

The roots of the secular equation (2.115) when $\omega \rightarrow \infty$ are given by three finite roots $w_1 = \tilde{W}_1, w_2 = \tilde{W}_2$ and $w_3 = \tilde{W}_3 \equiv \frac{\tilde{\sigma}}{\alpha_1 \varepsilon^{\frac{1}{2}} + \alpha_0 \tilde{\sigma}}$, and one infinite root. Now for ω sufficiently large power series expansions take the following form

$$w_i(\omega) = \tilde{W}_i + \sum_{n=1}^{\infty} d_n^{(i)}(-i\omega)^{(-n)}, \quad i = 1, 2, 3. \quad (2.126)$$

For stability we require $d_n^{(i)} < 0$, $i = 1, 2, 3$.

When $i = 1, n = 1$ we get

$$w_1(\omega) = \tilde{W}_1 + d_1^{(1)}(-i\omega)^{(-1)} + O(\omega^{-2}). \quad (2.127)$$

Substituting (2.127) into (2.115) we get

$$d_1^{(1)} = \frac{-\tilde{W}_1(\tilde{W}_1 - \tilde{p}_1)(\tilde{W}_1 - \tilde{p}_2)(\tilde{W}_1 - \tilde{p}_3)}{\tilde{\sigma}(\tilde{W}_1 - \tilde{W}_2)[\tilde{W}_1(\alpha_1\varepsilon^{\frac{1}{2}} + \alpha_0\tilde{\sigma}) - \tilde{\sigma}]}. \quad (2.128)$$

The sign of $d_1^{(1)}$ depends on the sign of $[\tilde{W}_1(\alpha_1\varepsilon^{\frac{1}{2}} + \alpha_0\tilde{\sigma}) - \tilde{\sigma}]$ and it is unaffected by the sign of \tilde{p}_1 . Stability is satisfied when

$$[\tilde{W}_1(\alpha_1\varepsilon^{\frac{1}{2}} + \alpha_0\tilde{\sigma}) - \tilde{\sigma}] < 0.$$

Therefore, if $[\tilde{W}_1(\alpha_1\varepsilon^{\frac{1}{2}} + \alpha_0\tilde{\sigma}) - \tilde{\sigma}] > 0$ then $d_1^{(1)} > 0$ and $w_1(\omega)$ is unstable. Insert (2.128) into (2.127) to obtain

$$w_1(\omega) = \tilde{W}_1 + i\left\{\omega\tilde{\sigma}[\tilde{W}_1(\alpha_1\varepsilon^{\frac{1}{2}} + \alpha_0\tilde{\sigma}) - \tilde{\sigma}]\right\}^{-1} \frac{\tilde{F}(\tilde{W}_1)}{\tilde{G}'(\tilde{W}_1)} + O(\omega^{-2}). \quad (2.129)$$

When $i = 2, n = 1$ we get

$$w_2(\omega) = \tilde{W}_2 + d_1^{(2)}(-i\omega)^{(-1)} + O(\omega^{-2}). \quad (2.130)$$

Substituting (2.130) into (2.115) we get

$$d_1^{(2)} = \frac{-\tilde{W}_2(\tilde{W}_2 - \tilde{p}_1)(\tilde{W}_2 - \tilde{p}_2)(\tilde{W}_2 - \tilde{p}_3)}{\tilde{\sigma}(\tilde{W}_2 - \tilde{W}_1)[\tilde{W}_2(\alpha_1\varepsilon^{\frac{1}{2}} + \alpha_0\tilde{\sigma}) - \tilde{\sigma}]}. \quad (2.131)$$

Again, $d_1^{(2)}$ is negative when $\tilde{W}_2(\alpha_1\varepsilon^{\frac{1}{2}} + \alpha_0\tilde{\sigma}) - \tilde{\sigma}$ is negative and either \tilde{p}_1 is negative or positive. Insert (2.131) into (2.130) to get

$$w_2(\omega) = \tilde{W}_2 + i\left\{\omega\tilde{\sigma}[\tilde{W}_2(\alpha_1\varepsilon^{\frac{1}{2}} + \alpha_0\tilde{\sigma}) - \tilde{\sigma}]\right\}^{-1} \frac{\tilde{F}(\tilde{W}_2)}{\tilde{G}'(\tilde{W}_2)} + O(\omega^{-2}). \quad (2.132)$$

$w_2(\omega)$ is stable when

$$\tilde{W}_2(\alpha_1\varepsilon^{\frac{1}{2}} + \alpha_0\tilde{\sigma}) - \tilde{\sigma} < 0 \quad \Rightarrow \quad \tilde{W}_2 < \frac{\tilde{\sigma}}{(\alpha_1\varepsilon^{\frac{1}{2}} + \alpha_0\tilde{\sigma})}.$$

When $i = 3, n = 1$

$$w_3(\omega) = \tilde{W}_3 + d_1^{(3)}(-i\omega)^{(-1)} + O(\omega^{-2}), \quad (2.133)$$

in which, as before,

$$\tilde{W}_3 = \frac{\tilde{\sigma}}{(\alpha_1 \varepsilon^{\frac{1}{2}} + \alpha_0 \tilde{\sigma})}. \quad (2.133a)$$

Substituting (2.133) into the secular equation (2.115) we obtain

$$d_1^{(3)} = \frac{-\tilde{F}(\tilde{W}_3)}{\tilde{\sigma}(\alpha_1 \varepsilon^{\frac{1}{2}} + \alpha_0 \tilde{\sigma})\tilde{G}(\tilde{W}_3)},$$

the sign of which is not easy to determine. We can determine the sign of $d_1^{(3)}$ in two special cases, firstly, as $\tilde{\sigma} \rightarrow 0$, a purely thermal constraint, and secondly, as $\tilde{\sigma} \rightarrow \infty$, a purely mechanical constraint. When $\tilde{\sigma} \rightarrow 0$, then $d_1^{(3)} \rightarrow \infty$, so

$$d_1^{(3)} = \frac{-\tilde{W}_3(\tilde{W}_3 - \tilde{p}_1)(\tilde{W}_3 - \tilde{p}_2)(\tilde{W}_3 - \tilde{p}_3)}{\tilde{\sigma}(\alpha_1 \varepsilon^{\frac{1}{2}} + \alpha_0 \tilde{\sigma})(\tilde{W}_3 - \tilde{W}_1)(\tilde{W}_3 - \tilde{W}_2)}. \quad (2.134)$$

Simplifying (2.134) with aid of (2.133a) leads to

$$d_1^{(3)} = \frac{-(\tilde{W}_3 - \tilde{p}_1)(\tilde{W}_3 - \tilde{p}_2)(\tilde{W}_3 - \tilde{p}_3)}{(\alpha_1 \varepsilon^{\frac{1}{2}} + \alpha_0 \tilde{\sigma})^2(\tilde{W}_3 - \tilde{W}_1)(\tilde{W}_3 - \tilde{W}_2)}.$$

As $\tilde{\sigma} \rightarrow 0$, $\tilde{W}_3 \rightarrow 0$, so

$$d_1^{(3)} = \frac{\tilde{p}_1 \tilde{p}_2 \tilde{p}_3}{(\alpha_1 \varepsilon^{\frac{1}{2}})^2 \tilde{W}_1 \tilde{W}_2}.$$

It is clear that if $\tilde{p}_1 > 0$, w_3 is unstable. But if $\tilde{p}_1 < 0$, w_3 becomes stable.

As $\tilde{\sigma} \rightarrow \infty$, $\tilde{W}_3 \rightarrow \alpha_0^{-1}$, then $d_1^{(3)} \rightarrow 0$ and higher powers of $(-i\omega)^{-1}$ are needed in the expansion (2.133).

The fourth root, which is large if ω is large, may be written as

$$w_4(\omega) = (i\omega)A + B + O(\omega^{-1}). \quad (2.135)$$

Inserting (2.135) into the secular equation (2.115) we get

$$(i\omega A + B)(i\omega A + B - \tilde{p}_1)(i\omega A + B - \tilde{p}_2)(i\omega A + B - \tilde{p}_3) - i\omega \tilde{\sigma} \left\{ [(i\omega A + B)(\alpha_1 \varepsilon^{\frac{1}{2}} + \alpha_0 \tilde{\sigma}) - \tilde{\sigma}] (i\omega A + B - \tilde{W}_1)(i\omega A + B - \tilde{W}_2) \right\} = 0. \quad (2.136)$$

Multiplying this equation by $(i\omega)^{-4}$ we get

$$(A + B(i\omega)^{-1})(A + (B - \tilde{p}_1)(i\omega)^{-1})(A + (B - \tilde{p}_2)(i\omega)^{-1})(A + (B - \tilde{p}_3)(i\omega)^{-1}) - \tilde{\sigma} \left\{ [(A + B(i\omega)^{-1})(\alpha_1 \varepsilon^{\frac{1}{2}} + \alpha_0 \tilde{\sigma}) - \tilde{\sigma}(i\omega)^{-1}][A + (B - \tilde{W}_1)(i\omega)^{-1}][A + (B - \tilde{W}_2)(i\omega)^{-1}] \right\} = 0. \quad (2.137)$$

For large ω we obtain

$$A^4 - \tilde{\sigma}A^3(\alpha_1\varepsilon^{\frac{1}{2}} + \alpha_0\tilde{\sigma}) = 0,$$

so that

$$A = 0, 0, 0, \tilde{\sigma}(\alpha_1\varepsilon^{\frac{1}{2}} + \alpha_0\tilde{\sigma}). \quad (2.138)$$

The three zero roots correspond to the roots already found. Expand (2.137):

$$A^3 \left\{ A + [(B - \tilde{p}_3)(i\omega)^{-1} + (B - \tilde{p}_2)(i\omega)^{-1} + (B - \tilde{p}_1)(i\omega)^{-1} + B(i\omega)^{-1}] \right\} - A^2 \tilde{\sigma} \left\{ (A + B(i\omega)^{-1}(\alpha_1\varepsilon^{\frac{1}{2}} + \alpha_0\tilde{\sigma}) - \tilde{\sigma}(i\omega)^{-1} + (B - \tilde{W}_2)(\alpha_1\varepsilon^{\frac{1}{2}} + \alpha_0\tilde{\sigma})(i\omega)^{-1} + (B - \tilde{W}_1)(\alpha_1\varepsilon^{\frac{1}{2}} + \alpha_0\tilde{\sigma})(i\omega)^{-1}) \right\} = 0 \quad (2.139)$$

Cancel A^2 , $A \neq 0$, to give

$$A \left\{ A + [(B - \tilde{p}_3)(i\omega)^{-1} + (B - \tilde{p}_2)(i\omega)^{-1} + (B - \tilde{p}_1)(i\omega)^{-1} + B(i\omega)^{-1}] \right\} - \tilde{\sigma} \left\{ (A + B(i\omega)^{-1}(\alpha_1\varepsilon^{\frac{1}{2}} + \alpha_0\tilde{\sigma}) - \tilde{\sigma}(i\omega)^{-1} + (B - \tilde{W}_2)(\alpha_1\varepsilon^{\frac{1}{2}} + \alpha_0\tilde{\sigma})(i\omega)^{-1} + (B - \tilde{W}_1)(\alpha_1\varepsilon^{\frac{1}{2}} + \alpha_0\tilde{\sigma})(i\omega)^{-1}) \right\} = 0. \quad (2.140)$$

The coefficient of $(i\omega)^{-1}$ must vanish:

$$A[4B - (\tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3)] - \tilde{\sigma}(\alpha_1\varepsilon^{\frac{1}{2}} + \alpha_0\tilde{\sigma})\{3B - (\tilde{W}_1 + \tilde{W}_2)\} + \tilde{\sigma}^2 = 0. \quad (2.141)$$

Using (2.138)₄ to give A we can find B :

$$B = \tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 - (\tilde{W}_1 + \tilde{W}_2) - \frac{\tilde{\sigma}}{(\alpha_1\varepsilon^{\frac{1}{2}} + \alpha_0\tilde{\sigma})}. \quad (2.142)$$

Thus,

$$w_4(\omega) = i\omega\tilde{\sigma}(\alpha_1\varepsilon^{\frac{1}{2}} + \alpha_0\tilde{\sigma}) + \tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 - (\tilde{W}_1 + \tilde{W}_2) - \frac{\tilde{\sigma}}{(\alpha_1\varepsilon^{\frac{1}{2}} + \alpha_0\tilde{\sigma})} + O(\omega^{-1}). \quad (2.143)$$

The stability condition is not satisfied here: $\text{Im } w_4(\omega) > 0$, so $w_4(\omega)$ is unstable.

From the secular equation (2.115), and previous arguments, we see that a branch cannot change from stable to unstable, or vice versa, for intermediate frequencies $0 < \omega < \infty$. This is borne out by Figures 2.6–2.8.

The exceptional case $\tilde{p}_1 = 0$.

When $\tilde{p}_1 = 0$ the secular equation (2.115), with the aid of (2.116), becomes

$$w^2(w - \tilde{p}_2)(w - \tilde{p}_3) - i\omega\tilde{\sigma}[w(\alpha_1\varepsilon^{\frac{1}{2}} + \alpha_0\tilde{\sigma}) - \tilde{\sigma}](w - \tilde{W}_1)(w - \tilde{W}_2) = 0.$$

For low frequencies we try the balance $w = A(-i\omega)^n$ and substitute into this secular equation in order to determine n . We find that $n = 1/2$ and then w is given by

$$w = \pm \left(-i\omega\tilde{\sigma}^2 \frac{\tilde{W}_1\tilde{W}_2}{\tilde{p}_2\tilde{p}_3} \right)^{\frac{1}{2}}.$$

These two branches begin at the origin and have arguments $-\pi/4$ and $3\pi/4$ in the complex w plane. This can be seen in Figure 2.8.

Numerical results

In Figure 2.6 we have taken $\tilde{p}_1 > 0$. The branch $w_0(\omega)$ beginning at the origin is unstable in each part of the Figure. All the other branches begin to the right of this branch. If α_0 and α_1 are small enough then

$$\tilde{p}_i(\alpha_1\varepsilon^{\frac{1}{2}} + \alpha_0\tilde{\sigma}) - \tilde{\sigma} < 0, \quad \text{for } i = 1, 2, 3,$$

and

$$\tilde{W}_i(\alpha_1\varepsilon^{1/2} + \alpha_0\tilde{\sigma}) - \tilde{\sigma} < 0, \quad \text{for } i = 1, 2$$

and so all the branches $w_i(\omega)$, $i = 1, 2, 3$, are stable. This can be seen in the first subfigures of Figure 2.6 where α_0 and α_1 are small. As α_0 and α_1 increase, first $w_3(\omega)$ becomes unstable, see part (d), and as they increase further other branches become unstable.

In Figure 2.7 we have taken $\tilde{p}_1 < 0$. The branch $w_1(\omega)$ beginning at $w = \tilde{p}_1$ is unstable in each part of the Figure. All the other branches begin to the right of this branch. The branch $w_0(\omega)$ begins at the origin and is stable in each part of the Figure. As in Figure 2.6, increasing α_0 and α_1 leads to increasing instability.

In Figure 2.8 we illustrate the exceptional case $\tilde{p}_1 = 0$. Now two branches emanate from the origin, namely, $w_0(\omega)$ and $w_1(\omega)$, one stable and the other unstable, one with argument $-\pi/4$ and the other with argument $3\pi/4$. The same increasing instability with increasing α_0 and α_1 is observed.

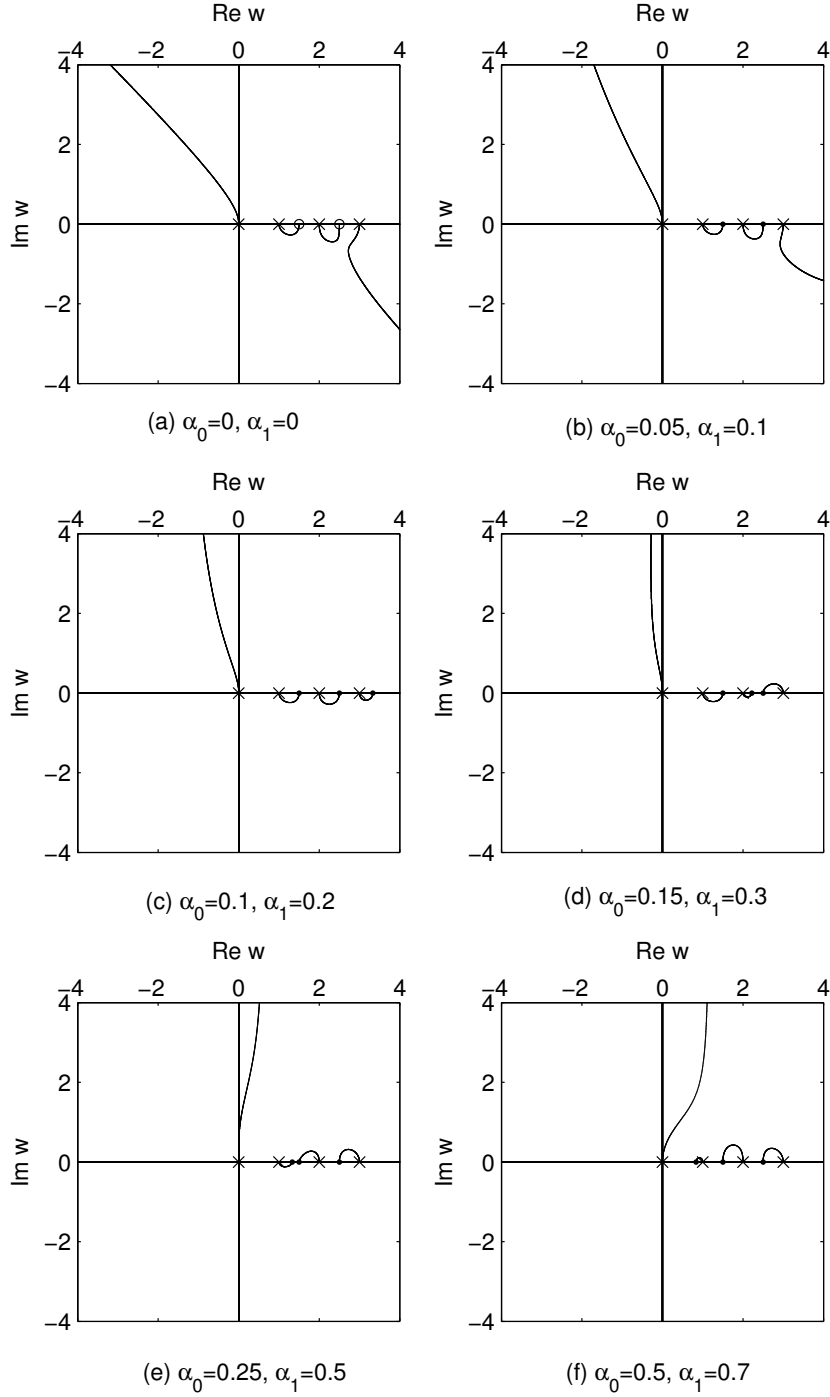


Figure 2.6: The longitudinal squared wave speeds of constrained anisotropic TRDTE theory. For each part, $\tilde{p}_1 = 1$, $\tilde{p}_2 = 2$, $\tilde{p}_3 = 3$, $\tilde{W}_1 = 1.5$, $\tilde{W}_2 = 2.5$, $\tilde{\sigma} = 1$, $\varepsilon = 1$.

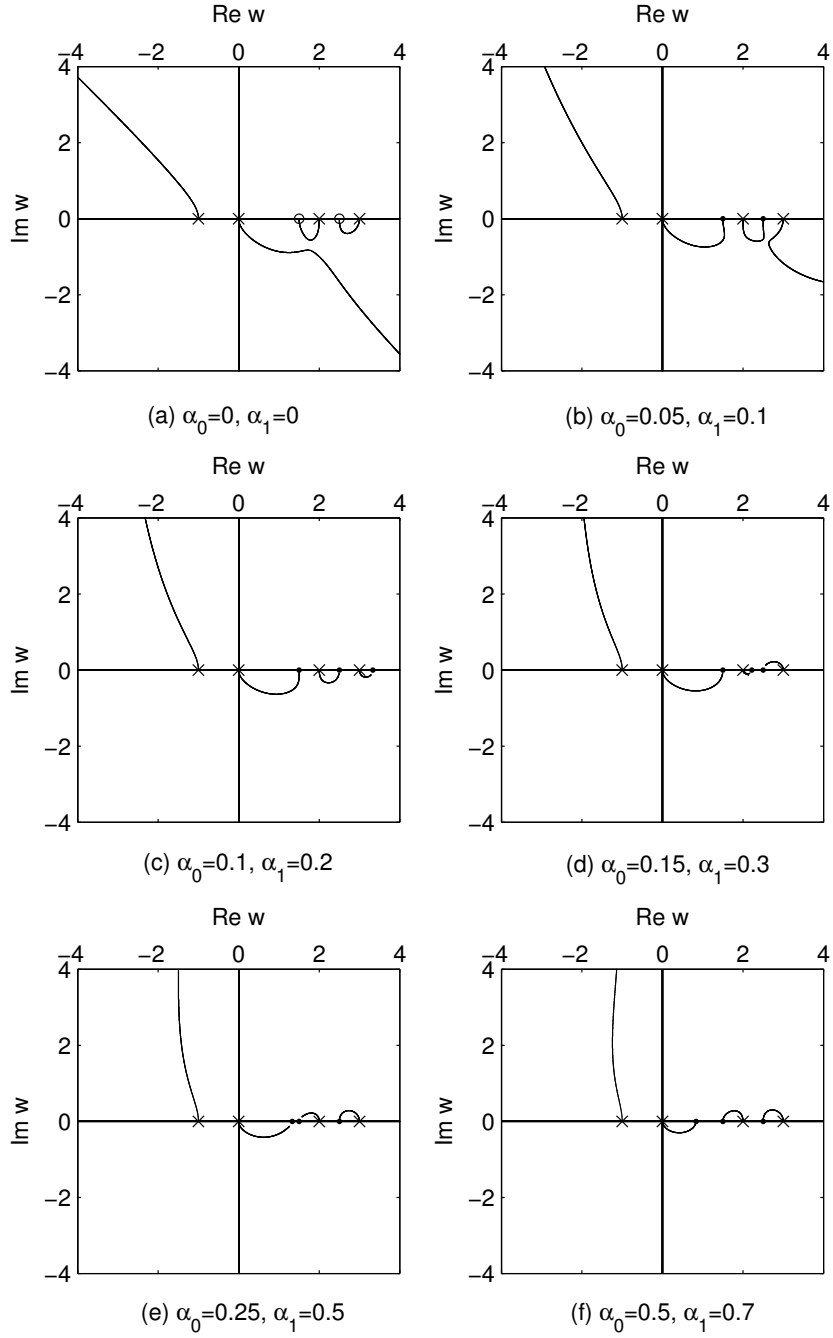


Figure 2.7: The longitudinal squared wave speeds of constrained anisotropic TRDTE theory. For each part, $\tilde{p}_1 = -1, \tilde{p}_2 = 2, \tilde{p}_3 = 3, \tilde{W}_1 = 1.5, \tilde{W}_2 = 2.5, \tilde{\sigma} = 1, \varepsilon = 1$.

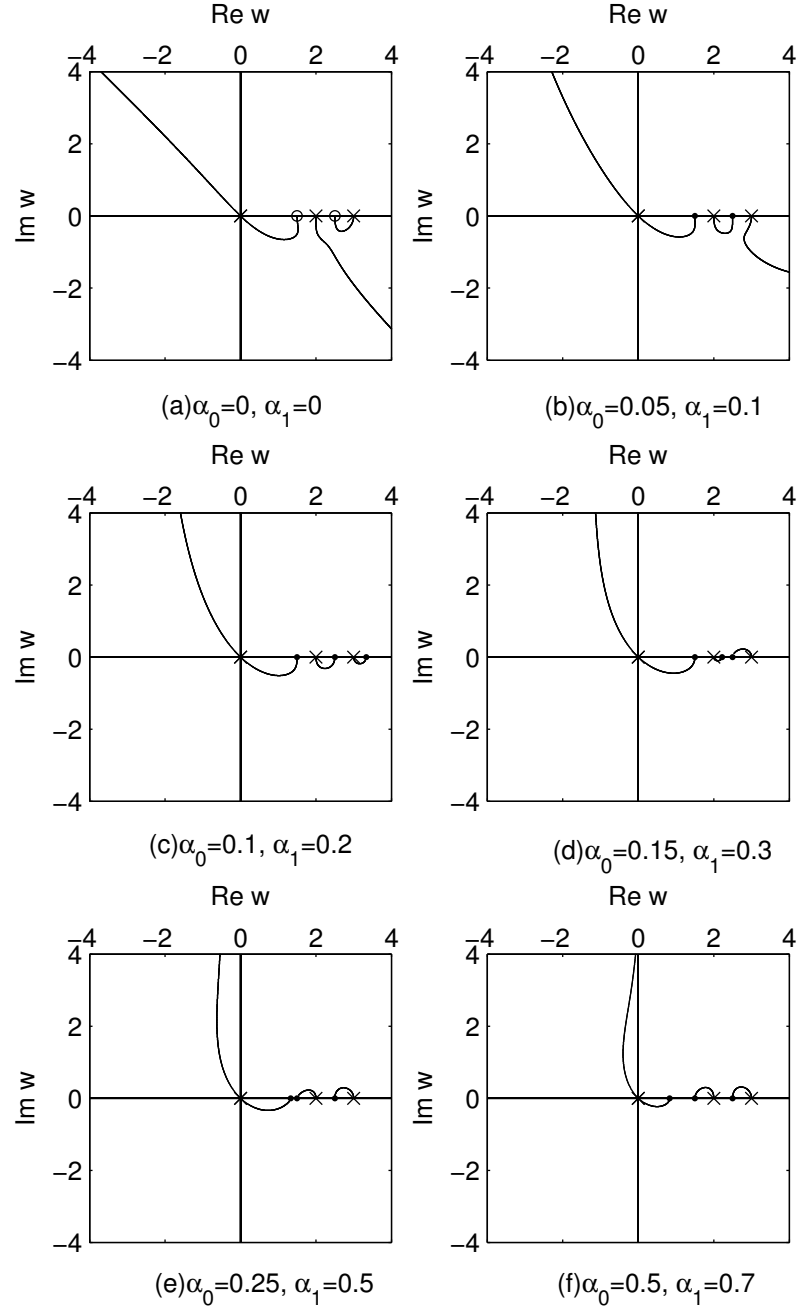


Figure 2.8: The longitudinal squared wave speeds of constrained anisotropic TRDTE theory. For each part, $\tilde{p}_1 = 0$, $\tilde{p}_2 = 2$, $\tilde{p}_3 = 3$, $\tilde{W}_1 = 1.5$, $\tilde{W}_2 = 2.5$, $\tilde{\sigma} = 1$, $\varepsilon = 1$.

2.3.2 Alternative form of deformation-temperature constraint

In this section we are concerned with using an alternative thermomechanical constraint relating deformation to temperature as shown in the following formula

$$\tilde{N}_{pq}u_{p,q} - \alpha(\theta + \alpha_0\dot{\theta}) = 0. \quad (2.144)$$

To explain the choice of the new form (2.144) of the deformation-temperature constraint we follow Scott [31]. We must first consider the justification of the form of constraint (2.81) in classical thermoelasticity. In classical thermoelasticity the Helmholtz free energy $\psi(e_{ij}, \theta)$ acts as a potential for the stress and entropy:

$$\sigma_{ij} = \rho \frac{\partial \psi}{\partial e_{ij}}, \quad \phi = -\frac{\partial \psi}{\partial \theta}, \quad (2.144a)$$

see, for example, [30, (2.6)]. To account for the constraint, a quantity

$$\rho^{-1}\tilde{\eta}(\tilde{N}_{pq}u_{p,q} - \alpha\theta),$$

see [8], where $\tilde{\eta}$ is a Lagrange multiplier, is added to ψ and then the differentiations in (2.144a) give the additional constraint stress $\tilde{\eta}\tilde{\mathbf{N}}$ and the additional constraint entropy $\rho^{-1}\alpha\tilde{\eta}$ occurring in (2.82). In their theory of temperature rate dependent thermoelasticity Green and Lindsay [21] introduce a second temperature function $\theta_1(\theta, \dot{\theta})$ which plays an important role in entropy production (and is denoted by ϕ in [21]). They show that (2.144a) must be replaced by

$$\sigma_{ij} = \rho \frac{\partial \psi}{\partial e_{ij}}, \quad \phi = -\frac{\partial \psi}{\partial \dot{\theta}} \bigg/ \frac{\partial \theta_1}{\partial \dot{\theta}}, \quad (2.144b)$$

see [21, Equations (3.15)₁ and (3.6), respectively]. We shall replace the constraint (2.81) of the classical theory by

$$\tilde{N}_{pq}u_{p,q} - \alpha\theta_1 = 0, \quad (2.144c)$$

in which the usual temperature θ has been replaced by the second temperature θ_1 . We adopt the linearised form [21, (4.4)] of θ_1 :

$$\theta_1 = \theta + \alpha_0\dot{\theta}, \quad (2.144d)$$

so that (2.144) is obtained immediately. To account for the new constraint, a quantity $\rho^{-1}\tilde{\eta}(\tilde{N}_{pq}u_{p,q} - \alpha\theta_1)$, where $\tilde{\eta}$ is again a Lagrange multiplier, is added to ψ and then the differentiations (2.144b), together with the linearisation (2.144d), give the additional constraint stress $\tilde{\eta}\tilde{\mathbf{N}}$ and the additional constraint entropy $\rho^{-1}\alpha\tilde{\eta}$ exactly as occurred in (2.82).

It follows that the other field equations of the anisotropic case (2.83) and (2.85) also hold in the present case:

$$\left. \begin{aligned} \tilde{c}_{ijkl}u_{k,jl} - \beta_{ij}(\theta_{,j} + \alpha_1\dot{\theta}_{,j}) + \tilde{N}_{ij}\tilde{\eta}_{,j} &= \rho\ddot{u}_i, \\ T\beta_{pq}\dot{u}_{p,q} + \rho c(\dot{\theta} + \alpha_0\ddot{\theta}) + T\alpha\dot{\tilde{\eta}} - k_{ij}\theta_{,ij} &= 0. \end{aligned} \right\} \quad (2.145)$$

The secular equation

Now we follow the same steps as in Section 2.3.1 to get the secular equation. Firstly, we look for solutions of equations (2.144) and (2.145) in the form of the plane harmonic waves (2.86) by inserting (2.86) into (2.144) and (2.145) to get

$$\left. \begin{aligned} (\tilde{Q}_{ik} - \rho s^{-2}\delta_{ik})U_k + i(\omega s)^{-1}[b_i(1 - i\omega\alpha_1)\Theta - \tilde{c}_i\tilde{H}] &= 0, \\ \omega s T b_p U_p - i\alpha T \tilde{H} + (\omega s^2 k - i\rho c(1 - i\omega\alpha_0))\Theta &= 0, \\ i\omega s \tilde{c}_p U_p - \alpha(1 - i\omega\alpha_0)\Theta &= 0, \end{aligned} \right\} \quad (2.146)$$

where $\tilde{N}_{pq}n_q = \tilde{c}_p$, $\beta_{ij}n_j = b_i$, $k_{ij}n_j = k$. We note that equations (2.146)_{1,2} are similar to (2.89) and (2.91). We must now eliminate Θ and \tilde{H} between (2.146). Firstly, we write equation (2.146)₃ with Θ as subject:

$$\Theta = \frac{i\omega s \tilde{c}_p U_p}{\alpha(1 - i\omega\alpha_0)}. \quad (2.147)$$

Substituting (2.147) into (2.146)₂ we get

$$\tilde{H} = -i\alpha^{-1}\omega s b_p U_p + (\alpha^2 T)^{-1}(1 - i\omega\alpha_0)^{-1}\omega s \tilde{c}_p U_p (\omega s^2 k - i\rho c(1 - i\omega\alpha_0)). \quad (2.148)$$

Inserting (2.147) and (2.148) into (2.146)₁ we obtain

$$\left\{ (\tilde{Q}_{ip} - \rho s^{-2}\delta_{ip}) - \alpha^{-1}(1 - i\omega\alpha_0)^{-1} \left(b_i \tilde{c}_p (1 - i\omega\alpha_1) + \tilde{c}_i b_p (1 - i\omega\alpha_0) \right) \right. \\ \left. - i(\alpha^2 T)^{-1}(1 - i\omega\alpha_0)^{-1}(\omega s^2 k - i\rho c(1 - i\omega\alpha_0))\tilde{c}_i \tilde{c}_p \right\} U_p = 0. \quad (2.149)$$

Expanding this equation we get

$$\left\{ \tilde{Q}_{ip} - \alpha^{-1}(1 - i\omega\alpha_0)^{-1}(b_i\tilde{c}_p + \tilde{c}_i b_p) - (\alpha^2 T)^{-1}(1 - i\omega\alpha_0)^{-1}(\rho c(1 - i\omega\alpha_0) + i\omega s^2 k) \tilde{c}_i \tilde{c}_p \right. \\ \left. + i\omega\alpha^{-1}(1 - i\omega\alpha_0)^{-1}(\alpha_1 b_i \tilde{c}_p + \alpha_0 \tilde{c}_i b_p) - \rho s^{-2} \delta_{ip} \right\} U_p = 0. \quad (2.150)$$

Rearranging this equation we get

$$\left\{ \tilde{Q}_{ip} - \rho s^{-2} \delta_{ip} - (1 - i\omega\alpha_0)^{-1} \left[\alpha^{-1}(b_i \tilde{c}_p + \tilde{c}_i b_p) + (\alpha^2 T)^{-1} \rho c \tilde{c}_i \tilde{c}_p \right] \right. \\ \left. + i\omega(1 - i\omega\alpha_0)^{-1} \left[\alpha^{-1}(\alpha_1 b_i \tilde{c}_p + \alpha_0 \tilde{c}_i b_p) - (\alpha^2 T)^{-1}(s^2 k - \rho c \alpha_0) \tilde{c}_p \tilde{c}_i \right] \right\} U_p = 0.$$

The non-zero amplitudes U_p satisfy (2.146) if and only if

$$\det \left\{ \tilde{\mathbf{Q}} - w\mathbf{1} - (1 - i\omega\alpha_0)^{-1} \left[\alpha^{-1}(\mathbf{b} \otimes \tilde{\mathbf{c}} + \tilde{\mathbf{c}} \otimes \mathbf{b}) + (\alpha^2 T)^{-1} \rho c \tilde{\mathbf{c}} \otimes \tilde{\mathbf{c}} \right] \right. \\ \left. + i\omega(1 - i\omega\alpha_0)^{-1} \left[\alpha^{-1}(\alpha_1 \mathbf{b} \otimes \tilde{\mathbf{c}} + \alpha_0 \tilde{\mathbf{c}} \otimes \mathbf{b}) - (\alpha^2 T)^{-1}(s^2 k - \rho c \alpha_0) \tilde{\mathbf{c}} \otimes \tilde{\mathbf{c}} \right] \right\} = 0. \quad (2.151)$$

On defining

$$\tilde{\mathbf{S}} := \tilde{\mathbf{Q}} - (1 - i\omega\alpha_0)^{-1} \left[\alpha^{-1}(\mathbf{b} \otimes \tilde{\mathbf{c}} + \tilde{\mathbf{c}} \otimes \mathbf{b}) + \frac{\rho c}{\alpha^2 T} \tilde{\mathbf{c}} \otimes \tilde{\mathbf{c}} \right], \quad (2.152)$$

with aid of (2.152) equation (2.151) may be written as

$$\det \left\{ (\tilde{\mathbf{S}} - w\mathbf{1}) + \frac{i\omega}{(1 - i\omega\alpha_0)} \left[\frac{\alpha_0}{\alpha} \tilde{\mathbf{c}} \otimes \mathbf{b} + \frac{\alpha_1}{\alpha} \mathbf{b} \otimes \tilde{\mathbf{c}} - \frac{(s^2 k - \rho c \alpha_0)}{\alpha^2 T} \tilde{\mathbf{c}} \otimes \tilde{\mathbf{c}} \right] \right\} = 0. \quad (2.153)$$

From definitions (2.110a) and (2.110b) we can rewrite equation (2.152) as

$$\tilde{\mathbf{S}} = \tilde{\mathbf{Q}} - (1 - i\omega\alpha_0)^{-1} \left[2\alpha^{-1} \beta + \rho c / \alpha^2 T \right] \mathbf{n} \otimes \mathbf{n}. \quad (2.154)$$

To non-dimensionlise (2.154) we need to use the dimensionless quantities (2.103), so equation (2.154) becomes after dropping dashes for convenience

$$\tilde{\mathbf{S}} = \tilde{\mathbf{Q}} - (1 - i\omega\alpha_0)^{-1} \tilde{\sigma} \left[2\varepsilon^{\frac{1}{2}} + \tilde{\sigma} \right] \mathbf{n} \otimes \mathbf{n}, \quad (2.155)$$

where

$$\tilde{\sigma} = c^{\frac{1}{2}} / \alpha. \quad (2.155a)$$

The dimensionless form of the secular equation (2.153), after dropping dashes, is

$$\det \left\{ (\tilde{\mathbf{S}} - w\mathbf{1}) + \frac{i\omega}{1 - i\omega\alpha_0} \left[\tilde{\sigma} \varepsilon^{\frac{1}{2}} (\alpha_0 \tilde{\mathbf{c}} \otimes \mathbf{b} + \alpha_1 \mathbf{b} \otimes \tilde{\mathbf{c}}) - \tilde{\sigma}^2 (w^{-1} - \alpha_0) \tilde{\mathbf{c}} \otimes \tilde{\mathbf{c}} \right] \right\} = 0. \quad (2.156)$$

By using the definition (2.155), and the fact that for incompressibility at uniform temperature with thermal isotropy we have

$$\mathbf{b} = \tilde{\mathbf{c}} = \mathbf{n}, \quad (2.157)$$

equation (2.156) may be rewritten as

$$\det \left\{ (\tilde{\mathbf{S}} - w\mathbf{1}) + \frac{i\omega}{1 - i\omega\alpha_0} \left[\tilde{\sigma}\varepsilon^{1/2}(\alpha_0\mathbf{n} \otimes \mathbf{n} + \alpha_1\mathbf{n} \otimes \mathbf{n}) - \tilde{\sigma}^2(w^{-1} - \alpha_0)\mathbf{n} \otimes \mathbf{n} \right] \right\} = 0. \quad (2.158)$$

On simplification the secular equation becomes

$$\det \left\{ (\tilde{\mathbf{S}} - w\mathbf{1}) + \frac{i\omega}{1 - i\omega\alpha_0} \left[\tilde{\sigma}\varepsilon^{1/2}(\alpha_0 + \alpha_1) - \tilde{\sigma}^2(w^{-1} - \alpha_0) \right] \mathbf{n} \otimes \mathbf{n} \right\} = 0. \quad (2.159)$$

By using the standard identity (2.60), equation (2.159) can be written as

$$\det(\tilde{\mathbf{S}} - w\mathbf{1}) + \frac{i\omega}{1 - i\omega\alpha_0} \left[\tilde{\sigma}\varepsilon^{1/2}(\alpha_0 + \alpha_1) - \tilde{\sigma}^2(w^{-1} - \alpha_0) \right] \mathbf{n} \cdot (\tilde{\mathbf{S}} - w\mathbf{1})^{\text{adj}} \mathbf{n} = 0. \quad (2.160)$$

Multiplying by w , equation (2.160) becomes

$$w \det(w\mathbf{1} - \tilde{\mathbf{S}}) - \frac{i\omega\tilde{\sigma}}{1 - i\omega\alpha_0} \left[w\varepsilon^{\frac{1}{2}}(\alpha_0 + \alpha_1) - \tilde{\sigma}(1 - w\alpha_0) \right] \mathbf{n} \cdot (w\mathbf{1} - \tilde{\mathbf{S}})^{\text{adj}} \mathbf{n} = 0. \quad (2.161)$$

We need to group the terms in w together to obtain

$$w \det(w\mathbf{1} - \tilde{\mathbf{S}}) - \frac{i\omega\tilde{\sigma}}{1 - i\omega\alpha_0} \left[w(\varepsilon^{\frac{1}{2}}(\alpha_0 + \alpha_1) + \tilde{\sigma}\alpha_0) - \tilde{\sigma} \right] \mathbf{n} \cdot (w\mathbf{1} - \tilde{\mathbf{S}})^{\text{adj}} \mathbf{n} = 0. \quad (2.162)$$

Now we need the secular equation (2.162) written in terms of $\tilde{\mathbf{P}}$ defined by (2.112)₂. The first term of (2.162) is

$$L_1 \equiv w \det(w\mathbf{1} - \tilde{\mathbf{S}}). \quad (2.162a)$$

With aid of the definition (2.155), we may rewrite (2.162a) as

$$L_1 = w \det \left\{ w\mathbf{1} - \tilde{\mathbf{Q}} + (2\tilde{\sigma}\varepsilon^{\frac{1}{2}} + \tilde{\sigma}^2)\mathbf{n} \otimes \mathbf{n} + \left(-1 + \frac{1}{1 - i\omega\alpha_0} \right) (2\tilde{\sigma}\varepsilon^{\frac{1}{2}} + \tilde{\sigma}^2)\mathbf{n} \otimes \mathbf{n} \right\}.$$

By simplifying L_1 becomes

$$L_1 = w \det \left\{ w\mathbf{1} - \tilde{\mathbf{P}} + \frac{i\omega\alpha_0}{1 - i\omega\alpha_0} (2\tilde{\sigma}\varepsilon^{\frac{1}{2}} + \tilde{\sigma}^2)\mathbf{n} \otimes \mathbf{n} \right\}. \quad (2.162b)$$

By using the standard identity (2.60), (2.162b) may written as

$$L_1 = w \det(w\mathbf{1} - \tilde{\mathbf{P}}) + \frac{i\omega\alpha_0 w}{1 - i\omega\alpha_0} (2\tilde{\sigma}\varepsilon^{\frac{1}{2}} + \tilde{\sigma}^2) \mathbf{n} \cdot (w\mathbf{1} - \tilde{\mathbf{P}})^{\text{adj}} \mathbf{n}. \quad (2.162c)$$

The last factor of the secular equation (2.162) is termed

$$L_2 \equiv \mathbf{n} \cdot (w\mathbf{1} - \tilde{\mathbf{P}})^{\text{adj}} \mathbf{n}. \quad (2.162d)$$

The expression (2.162d) may be written in terms of $\tilde{\mathbf{P}}$ similarly to (2.162a) as

$$L_2 = \mathbf{n} \cdot \left(w\mathbf{1} - \tilde{\mathbf{P}} + \left(\frac{i\omega\alpha_0}{1 - i\omega\alpha_0} \right) (2\tilde{\sigma}\varepsilon^{\frac{1}{2}} + \tilde{\sigma}^2) \mathbf{n} \otimes \mathbf{n} \right)^{\text{adj}} \mathbf{n}. \quad (2.162e)$$

By using [30, (A2)], we may rewrite (2.162e) as

$$L_2 = \mathbf{n} \cdot (w\mathbf{1} - \tilde{\mathbf{P}})^{\text{adj}} \mathbf{n}. \quad (2.162f)$$

By substituting (2.162c) and (2.162f) into the secular equation (2.162) we obtain

$$\begin{aligned} w \det(w\mathbf{1} - \tilde{\mathbf{P}}) + \frac{i\omega\alpha_0 w}{1 - i\omega\alpha_0} (2\tilde{\sigma}\varepsilon^{\frac{1}{2}} + \tilde{\sigma}^2) \mathbf{n} \cdot (w\mathbf{1} - \tilde{\mathbf{P}})^{\text{adj}} \mathbf{n} \\ - \frac{i\omega\tilde{\sigma}}{1 - i\omega\alpha_0} \left[w(\varepsilon^{\frac{1}{2}}(\alpha_0 + \alpha_1) + \tilde{\sigma}\alpha_0) - \tilde{\sigma} \right] \mathbf{n} \cdot (w\mathbf{1} - \tilde{\mathbf{P}})^{\text{adj}} \mathbf{n} = 0. \end{aligned} \quad (2.163)$$

Rearranging (2.163) gives

$$w \det(w\mathbf{1} - \tilde{\mathbf{P}}) + \frac{i\omega\tilde{\sigma}}{1 - i\omega\alpha_0} \left[w\alpha_0(2\varepsilon^{\frac{1}{2}} + \tilde{\sigma}) - w(\varepsilon^{\frac{1}{2}}\alpha_0 + \varepsilon^{\frac{1}{2}}\alpha_1 + \tilde{\sigma}\alpha_0) + \tilde{\sigma} \right] \mathbf{n} \cdot (w\mathbf{1} - \tilde{\mathbf{P}})^{\text{adj}} \mathbf{n} = 0. \quad (2.164)$$

Simplifying (2.164) we get

$$w \det(w\mathbf{1} - \tilde{\mathbf{P}}) + \frac{i\omega\tilde{\sigma}}{1 - i\omega\alpha_0} \left[w\varepsilon^{\frac{1}{2}}(\alpha_0 - \alpha_1) + \tilde{\sigma} \right] \mathbf{n} \cdot (w\mathbf{1} - \tilde{\mathbf{P}})^{\text{adj}} \mathbf{n} = 0. \quad (2.165)$$

Since $\alpha_1 > \alpha_0$, it is better to rewrite (2.165) as

$$w \det(w\mathbf{1} - \tilde{\mathbf{P}}) - \frac{i\omega\tilde{\sigma}}{1 - i\omega\alpha_0} \left[w\varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0) - \tilde{\sigma} \right] \mathbf{n} \cdot (w\mathbf{1} - \tilde{\mathbf{P}})^{\text{adj}} \mathbf{n} = 0. \quad (2.166)$$

This is the secular equation for anisotropic TRDTE which is constrained by the alternative deformation temperature constraint and has not previously appeared in the literature.

We already know that $\tilde{\mathbf{P}}$ has three real eigenvalues $\tilde{p}_i, i = 1, 2, 3$. Equation (2.166) written in terms of these eigenvalues is

$$\begin{aligned} w(w - \tilde{p}_1)(w - \tilde{p}_2)(w - \tilde{p}_3) - \frac{i\omega\tilde{\sigma}}{1 - i\omega\alpha_0} \left[w\varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0) - \tilde{\sigma} \right] \\ \{n_1^2(w - \tilde{p}_2)(w - \tilde{p}_3) + n_2^2(w - \tilde{p}_1)(w - \tilde{p}_3) + n_3^2(w - \tilde{p}_1)(w - \tilde{p}_2)\} = 0. \end{aligned} \quad (2.167)$$

Similarly to (2.114), the quadratic part within braces has two zeros \tilde{W}_1 and \tilde{W}_2 which satisfy

$$\tilde{p}_1 < \tilde{W}_1 < \tilde{p}_2 < \tilde{W}_2 < \tilde{p}_3. \quad (2.167a)$$

In a more simplified form the secular equation (2.167) is written as

$$\tilde{F}(w) - \frac{i\omega\tilde{\sigma}}{1 - i\omega\alpha_0} \left[w\varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0) - \tilde{\sigma} \right] \tilde{G}(w) = 0, \quad (2.168)$$

in which $\tilde{F}(w)$ and $\tilde{G}(w)$ are defined earlier in (2.116).

Similar to equation (2.115), by putting $\alpha_1 = \alpha_0 = 0$ we will recover the secular equation of anisotropic material which is constrained by deformation temperature constraint in classical thermoelasticity, see [30, (3.6)].

Low frequency expansions

When $\omega = 0$, the roots of the secular equation (2.168) are the zeros of $\tilde{F}(w)$, namely, \tilde{p}_i , $i = 0, 1, 2, 3$, defining $\tilde{p}_0 \equiv 0$. When $\omega \rightarrow 0$, the roots of the secular equation may be obtained by using Taylor expansions

$$w_i(\omega) = \tilde{p}_i + \sum_{n=1}^{\infty} d_n^{(i)} (-i\omega)^n, \quad i = 1, 2, 3. \quad (2.169)$$

When $i = 0, n = 1$, we get

$$w_0(\omega) = \tilde{p}_0 + d_1^{(0)}(-i\omega) + O(\omega^2). \quad (2.170)$$

Inserting (2.170) into (2.168) we get

$$d_1^{(0)} = -\tilde{\sigma}^2 \frac{\tilde{W}_1 \tilde{W}_2}{\tilde{p}_1 \tilde{p}_2 \tilde{p}_3}. \quad (2.171)$$

The sign of $d_1^{(0)}$ depends on the sign of \tilde{p}_1 .

If $\tilde{p}_1 < 0$ then $d_1^{(0)} > 0$ and so $w_0(\omega)$ is stable. But if $\tilde{p}_1 > 0$ then $d_1^{(0)} < 0$ and $w_0(\omega)$ is unstable. The exceptional case $\tilde{p}_1 = 0$ is considered later.

Substituting (2.171) into (2.170) we obtain

$$w_0(\omega) = i\omega\tilde{\sigma}^2 \frac{\tilde{W}_1 \tilde{W}_2}{\tilde{p}_1 \tilde{p}_2 \tilde{p}_3} + O(\omega^2).$$

When $i = 1, n = 1$, we get

$$w_1(\omega) = \tilde{p}_1 + d_1^{(1)}(-i\omega) + O(\omega^2) \quad (2.172)$$

and inserting (2.172) into (2.168) gives

$$d_1^{(1)} = -\tilde{\sigma}(\tilde{p}_1 \varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0) - \tilde{\sigma}) \frac{(\tilde{p}_1 - \tilde{W}_1)(\tilde{p}_1 - \tilde{W}_2)}{\tilde{p}_1(\tilde{p}_1 - \tilde{p}_2)(\tilde{p}_1 - \tilde{p}_3)}. \quad (2.173)$$

The sign of $d_1^{(1)}$ depends on the sign of $[\tilde{p}_1 \varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0) - \tilde{\sigma}]$ and \tilde{p}_1 . If $\tilde{p}_1 < 0$ then $d_1^{(1)} < 0$, so that $w_1(\omega)$ is unstable. On the other hand, $w_1(\omega)$ is stable when $d_1^{(1)} > 0$, and $d_1^{(1)}$ is positive only if

$$0 < \tilde{p}_1 < \frac{\tilde{\sigma}}{\varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0)}.$$

The exceptional case $\tilde{p}_1 = 0$ is considered later.

Substituting (2.173) into (2.172) we get

$$w_1(\omega) = \tilde{p}_1 + i\omega \tilde{\sigma}(\tilde{p}_1 \varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0) - \tilde{\sigma}) \frac{\tilde{G}(\tilde{p}_1)}{\tilde{F}'(\tilde{p}_1)} + O(\omega^2).$$

Similarly, when $i = 2, 3, n = 1$, we get

$$w_2(\omega) = \tilde{p}_2 + d_1^{(2)}(-i\omega) + O(\omega^2), \quad (2.174)$$

$$w_3(\omega) = \tilde{p}_3 + d_1^{(3)}(-i\omega) + O(\omega^2). \quad (2.175)$$

Insert (2.174) and (2.175) into (2.168) to get

$$d_1^{(2)} = -\tilde{\sigma}(\tilde{p}_2 \varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0) - \tilde{\sigma}) \frac{(\tilde{p}_2 - \tilde{W}_1)(\tilde{p}_2 - \tilde{W}_2)}{\tilde{p}_2(\tilde{p}_2 - \tilde{p}_1)(\tilde{p}_2 - \tilde{p}_3)}, \quad (2.176)$$

$$d_1^{(3)} = -\tilde{\sigma}(\tilde{p}_3 \varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0) - \tilde{\sigma}) \frac{(\tilde{p}_3 - \tilde{W}_1)(\tilde{p}_3 - \tilde{W}_2)}{\tilde{p}_3(\tilde{p}_3 - \tilde{p}_1)(\tilde{p}_3 - \tilde{p}_2)}. \quad (2.177)$$

The signs of $d_1^{(2)}$ and $d_1^{(3)}$ depend on the signs of $[\tilde{p}_2 \varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0) - \tilde{\sigma}]$ and $[\tilde{p}_3 \varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0) - \tilde{\sigma}]$, respectively, and are not affected by the sign of \tilde{p}_1 . Stability is satisfied if $d_1^{(2)}$ and $d_1^{(3)}$ are positive, and they are positive when

$$\tilde{p}_2 < \frac{\tilde{\sigma}}{\varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0)},$$

and

$$\tilde{p}_3 < \frac{\tilde{\sigma}}{\varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0)},$$

respectively. However, if $d_1^{(2)}$ and $d_1^{(3)}$ are negative, then $w_2(\omega)$ and $w_3(\omega)$ are unstable. Inserting (2.176) and (2.177) into (2.174) and (2.175), respectively, we get

$$w_2(\omega) = \tilde{p}_2 + i\omega\tilde{\sigma}(\tilde{p}_2\varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0) - \tilde{\sigma})\frac{\tilde{G}(\tilde{p}_2)}{\tilde{F}'(\tilde{p}_2)} + O(\omega^2),$$

$$w_3(\omega) = \tilde{p}_3 + i\omega\tilde{\sigma}(\tilde{p}_3\varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0) - \tilde{\sigma})\frac{\tilde{G}(\tilde{p}_3)}{\tilde{F}'(\tilde{p}_3)} + O(\omega^2).$$

Summarising, if $\tilde{p}_i\varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0) - \tilde{\sigma} < 0$ for any $i = 1, 2, 3$ then the corresponding branch $w_i(\omega)$ is stable. Conversely, if this quantity is positive the corresponding branch is unstable. The situation is like that described at the end of the low-frequency subsection of Section 2.3.1 involving instead the quantity $\tilde{p}_i(\alpha_1\varepsilon^{\frac{1}{2}} + \alpha_0\tilde{\sigma}) - \tilde{\sigma}$.

High frequency expansions

When $\omega \rightarrow \infty$, the secular equation (2.168) becomes

$$H(w) := \tilde{F}(w) + \frac{\tilde{\sigma}}{\alpha_0}(w\varepsilon^{1/2}(\alpha_1 - \alpha_0) - \tilde{\sigma})\tilde{G}(w) = 0. \quad (2.178)$$

$H(w)$ is a quartic in w , so there are four roots. The zeros of $H(w)$ are denoted by $\bar{h}_1, \bar{h}_2, \bar{h}_3, \bar{h}_4$. In order to find these zeros we need to examine the sign changes of $H(w)$. By using equation (2.178) and inequalities (2.167a) we find that

$$\begin{aligned} H(-\infty) &= \infty > 0, \\ H(0) &= -\alpha_0^{-1}\tilde{\sigma}^2\tilde{W}_1\tilde{W}_2 < 0, \\ H(\tilde{p}_1) &= \alpha_0^{-1}\tilde{\sigma}(\tilde{p}_1\varepsilon^{1/2}(\alpha_1 - \alpha_0) - \tilde{\sigma})(\tilde{p}_1 - \tilde{W}_1)(\tilde{p}_1 - \tilde{W}_2), \\ H(\tilde{W}_1) &= \tilde{W}_1(\tilde{W}_1 - \tilde{p}_1)(\tilde{W}_1 - \tilde{p}_2)(\tilde{W}_1 - \tilde{p}_3) > 0, \\ H(\tilde{p}_2) &= \alpha_0^{-1}\tilde{\sigma}(\tilde{p}_2\varepsilon^{1/2}(\alpha_1 - \alpha_0) - \tilde{\sigma})(\tilde{p}_2 - \tilde{W}_1)(\tilde{p}_2 - \tilde{W}_2), \\ H(\tilde{W}_2) &= \tilde{W}_2(\tilde{W}_2 - \tilde{p}_1)(\tilde{W}_2 - \tilde{p}_2)(\tilde{W}_2 - \tilde{p}_3) < 0, \\ H(\tilde{p}_3) &= \alpha_0^{-1}\tilde{\sigma}(\tilde{p}_3\varepsilon^{1/2}(\alpha_1 - \alpha_0) - \tilde{\sigma})(\tilde{p}_3 - \tilde{W}_1)(\tilde{p}_3 - \tilde{W}_2), \\ H(\infty) &= \infty > 0. \end{aligned} \quad (2.179)$$

From the inequalities (2.179), the zeros $\bar{h}_i, i = 1, 2, 3, 4$, of $H(w)$ interlace according to

$$\bar{h}_1 < 0 < \bar{h}_2 < \tilde{W}_1 < \bar{h}_3 < \tilde{W}_2 < \bar{h}_4. \quad (2.180)$$

Equation (2.178) may be written as

$$\tilde{F}(w) = H(w) - \alpha_0^{-1} \tilde{\sigma} (w\varepsilon^{1/2}(\alpha_1 - \alpha_0) - \tilde{\sigma}) \tilde{G}(w). \quad (2.181)$$

The secular equation (2.168) can be written as

$$\tilde{F}(w) = \frac{i\omega\tilde{\sigma}}{1 - i\omega\alpha_0} (w\varepsilon^{1/2}(\alpha_1 - \alpha_0) - \tilde{\sigma}) \tilde{G}(w). \quad (2.182)$$

By subtracting (2.182) from (2.181) we get the secular equation in the form

$$H(w) - \left(\frac{1}{\alpha_0} + \frac{i\omega}{1 - i\omega\alpha_0} \right) \tilde{\sigma} (w\varepsilon^{1/2}(\alpha_1 - \alpha_0) - \tilde{\sigma}) \tilde{G}(w) = 0. \quad (2.183)$$

Define a quartic polynomial

$$\bar{h}(w) = (w - \bar{h}_1)(w - \bar{h}_2)(w - \bar{h}_3)(w - \bar{h}_4), \quad (2.184)$$

which must be a scalar multiple of $H(w)$ because both have the same four roots. By comparing coefficients of w^4 we find that

$$H(w) = \bar{h}(w). \quad (2.185)$$

Insert (2.185) into (2.183) we get

$$\alpha_0(1 - i\omega\alpha_0)\bar{h}(w) - \tilde{\sigma} (w\varepsilon^{1/2}(\alpha_1 - \alpha_0) - \tilde{\sigma}) \tilde{G}(w) = 0, \quad (2.186)$$

an alternative form of the secular equation. Now, Taylor expansions of the roots of the secular equation (2.186) in the high frequency limit take the form

$$w_i(\omega) = \bar{h}_i + \sum_{n=1}^{\infty} d_n^{(i)} (-i\omega\alpha_0)^{-n}, \quad i = 1, 2, 3, 4, \quad (2.187)$$

so that $d_n^{(i)} < 0$ implies stability. When $i = 1, n = 1$, it may be shown that

$$w_1(\omega) = \bar{h}_1 + d_1^{(1)} (-i\omega\alpha_0)^{-1} + O(\omega^{-2}). \quad (2.188)$$

Substituting (2.188) into (2.186), and with aid of (2.180), we get

$$d_1^{(1)} = \alpha_0^{-1} \tilde{\sigma} (\bar{h}_1 \varepsilon^{1/2} (\alpha_1 - \alpha_0) - \tilde{\sigma}) \frac{(\bar{h}_1 - \tilde{W}_1)(\bar{h}_1 - \tilde{W}_2)}{(\bar{h}_1 - \bar{h}_2)(\bar{h}_1 - \bar{h}_3)(\bar{h}_1 - \bar{h}_4)} > 0,$$

as $\bar{h}_1 < 0$. So, the first branch is

$$w_1(\omega) = \bar{h}_1 + i\omega^{-1} \frac{\tilde{\sigma}}{\alpha_0^2} (\bar{h}_1 \varepsilon^{1/2} (\alpha_1 - \alpha_0) - \tilde{\sigma}) \frac{\tilde{G}(\bar{h}_1)}{\bar{h}'(\bar{h}_1)} + O(\omega^{-2}), \quad (2.189)$$

which is clearly unstable because $\text{Im } w_1(\omega) > 0$.

Similarly, when $i = 2, n = 1$ we obtain

$$d_1^{(2)} = \alpha_0^{-1} \tilde{\sigma} (\bar{h}_2 \varepsilon^{1/2} (\alpha_1 - \alpha_0) - \tilde{\sigma}) \frac{(\bar{h}_2 - \tilde{W}_1)(\bar{h}_2 - \tilde{W}_2)}{(\bar{h}_2 - \bar{h}_1)(\bar{h}_2 - \bar{h}_3)(\bar{h}_2 - \bar{h}_4)}.$$

We see that sign of $d_1^{(2)}$ depends on the sign of $[\bar{h}_2 \varepsilon^{1/2} (\alpha_1 - \alpha_0) - \tilde{\sigma}]$. If $d_1^{(2)} > 0$ then $w_2(\omega)$ is unstable, it is stable if $d_1^{(2)} < 0$, and $d_1^{(2)}$ is negative if

$$\bar{h}_2 \varepsilon^{1/2} (\alpha_1 - \alpha_0) - \tilde{\sigma} < 0,$$

i.e.

$$\bar{h}_2 < \frac{\tilde{\sigma}}{\varepsilon^{1/2} (\alpha_1 - \alpha_0)}.$$

Thus, we note that

$$w_2(\omega) = \bar{h}_2 + i\omega^{-1} \frac{\tilde{\sigma}}{\alpha_0^2} (\bar{h}_2 \varepsilon^{1/2} (\alpha_1 - \alpha_0) - \tilde{\sigma}) \frac{\tilde{G}(\bar{h}_2)}{\bar{h}'(\bar{h}_2)} + O(\omega^{-2}).$$

Similarly, when $i = 3, 4$ and $n = 1$

$$w_3(\omega) = \bar{h}_3 + i\omega^{-1} \frac{\tilde{\sigma}}{\alpha_0^2} (\bar{h}_3 \varepsilon^{1/2} (\alpha_1 - \alpha_1) - \tilde{\sigma}) \frac{\tilde{G}(\bar{h}_3)}{\bar{h}'(\bar{h}_3)} + O(\omega^{-2})$$

and $w_3(\omega)$ is stable when

$$\bar{h}_3 < \frac{\tilde{\sigma}}{\varepsilon^{1/2} (\alpha_1 - \alpha_0)}.$$

Also,

$$w_4(\omega) = \bar{h}_4 + i\omega^{-1} \frac{\tilde{\sigma}}{\alpha_0^2} (\bar{h}_4 \varepsilon^{1/2} (\alpha_1 - \alpha_0) - \tilde{\sigma}) \frac{\tilde{G}(\bar{h}_4)}{\bar{h}'(\bar{h}_4)} + O(\omega^{-2})$$

and so $w_4(\omega)$ is stable when

$$\bar{h}_4 < \frac{\tilde{\sigma}}{\varepsilon^{1/2} (\alpha_1 - \alpha_1)}.$$

Summarising, the stability of each branch is determined by the sign of

$$\bar{h}_i \varepsilon^{1/2} (\alpha_1 - \alpha_0) - \tilde{\sigma}, \quad i = 1, 2, 3, 4,$$

stable if negative, unstable if positive.

As before, the branches cannot change their stability for intermediate frequencies.

The exceptional case $\tilde{p}_1 = 0$.

When $\tilde{p}_1 = 0$ the situation is as at the end of Section 2.3.1 and we have two branches beginning at the origin with arguments $-\pi/4$ and $3\pi/4$ in the complex w plane, see Figure 2.10.

Numerical results

Figure 2.9 illustrates the case $\tilde{p}_1 > 0$. The branch $w_0(\omega)$ beginning at the origin is always unstable in each part of the Figure. If α_0 and α_1 are small enough then

$$\tilde{p}_i \varepsilon^{1/2} (\alpha_1 - \alpha_0) - \tilde{\sigma} < 0, \quad \text{for } i = 1, 2, 3,$$

$$\bar{h}_i \varepsilon^{1/2} (\alpha_1 - \alpha_0) - \tilde{\sigma} < 0, \quad \text{for } i = 1, 2, 3,$$

and so all the branches $w_i(\omega)$, $i = 1, 2, 3$, are stable. This is clear in the first subfigures (a)–(c) of Figure 2.9 where α_0 and α_1 are small. As α_0 and α_1 increase, first $w_3(\omega)$ becomes unstable, see part (d), and as they increase further other branches become unstable.

Figure 2.10 illustrates the exceptional case $\tilde{p}_1 = 0$ and we see two branches beginning at the origin, one stable with argument $-\pi/4$, the other unstable with argument $-3\pi/4$. As with Figure 2.9, more branches become unstable as α_0 and α_1 increase. The branch $w_0(\omega)$ begins at the origin and is stable in each part of the Figures.

Figure 2.11 illustrates the case $\tilde{p}_1 < 0$. The left hand branch is always unstable and, as with Figures 2.9 and 2.10, more branches become unstable as α_0 and α_1 increase. As with Figure 2.10, the branch $w_0(\omega)$ begins at the origin and is stable in each part of the Figure.

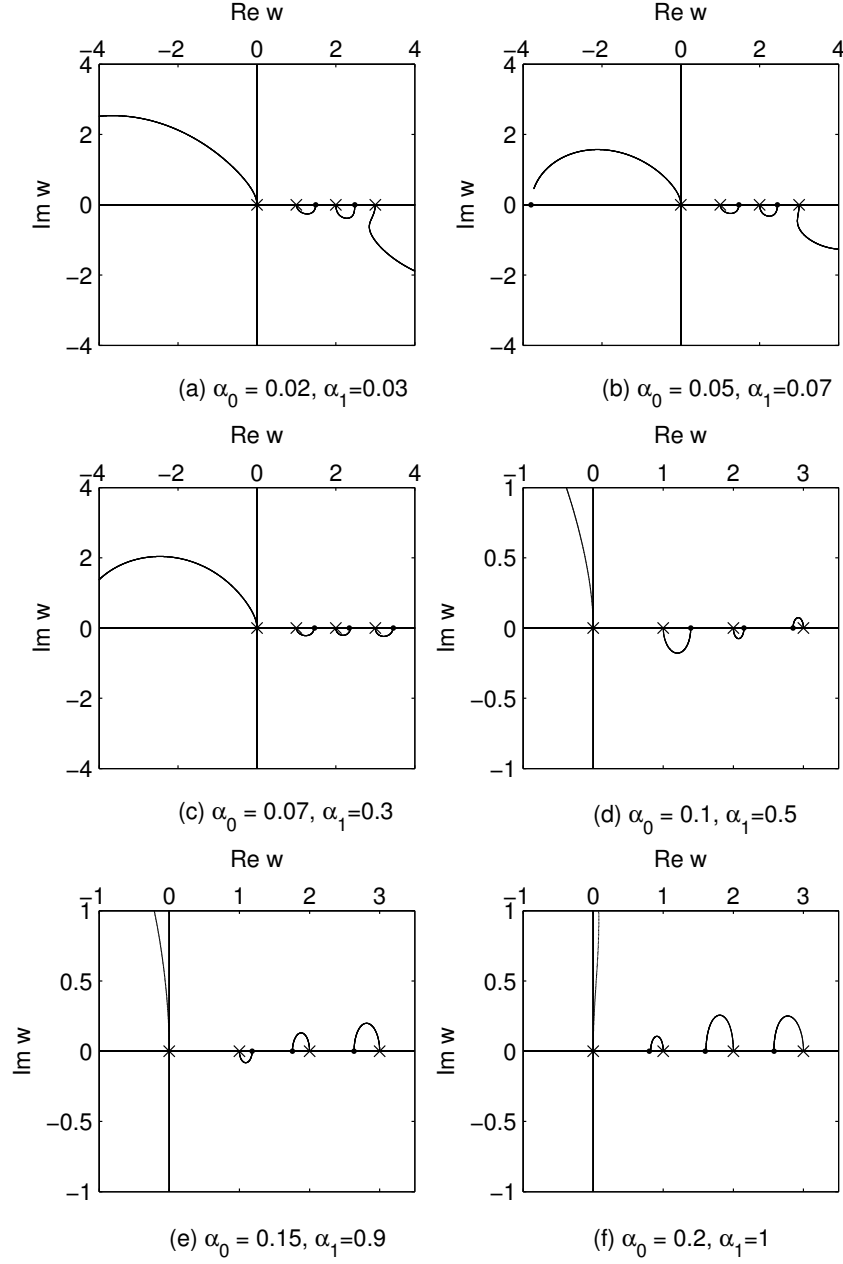


Figure 2.9: The longitudinal squared wave speeds of constrained anisotropic TRDTE theory. For each part, $\tilde{p}_1 = 1, \tilde{p}_2 = 2, \tilde{p}_3 = 3, \tilde{W}_1 = 1.5, \tilde{W}_2 = 2.5, \tilde{\sigma} = 1, \varepsilon = 1$.

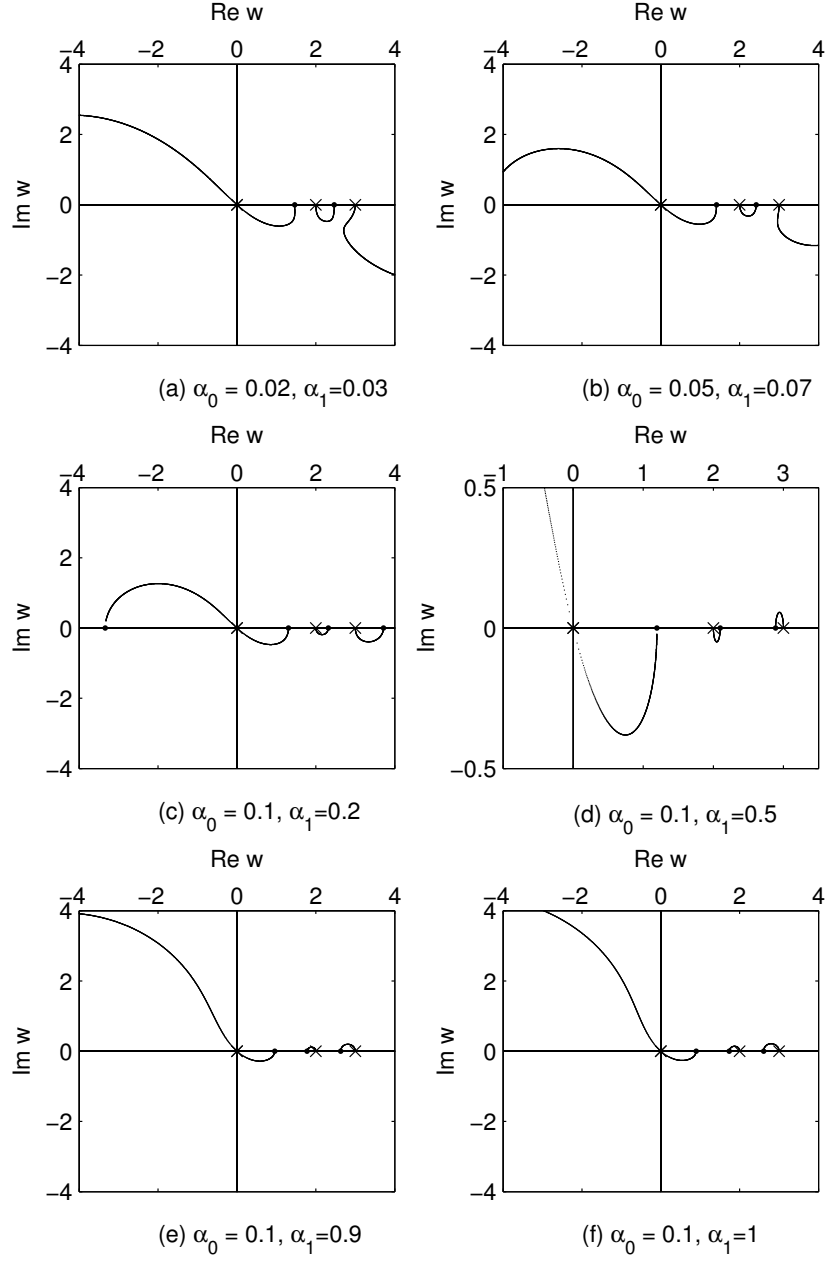


Figure 2.10: The longitudinal squared wave speeds of constrained anisotropic TRDTE theory. For each part, $\tilde{p}_1 = 0, \tilde{p}_2 = 2, \tilde{p}_3 = 3, \tilde{W}_1 = 1.5, \tilde{W}_2 = 2.5, \tilde{\sigma} = 1, \varepsilon = 1$.

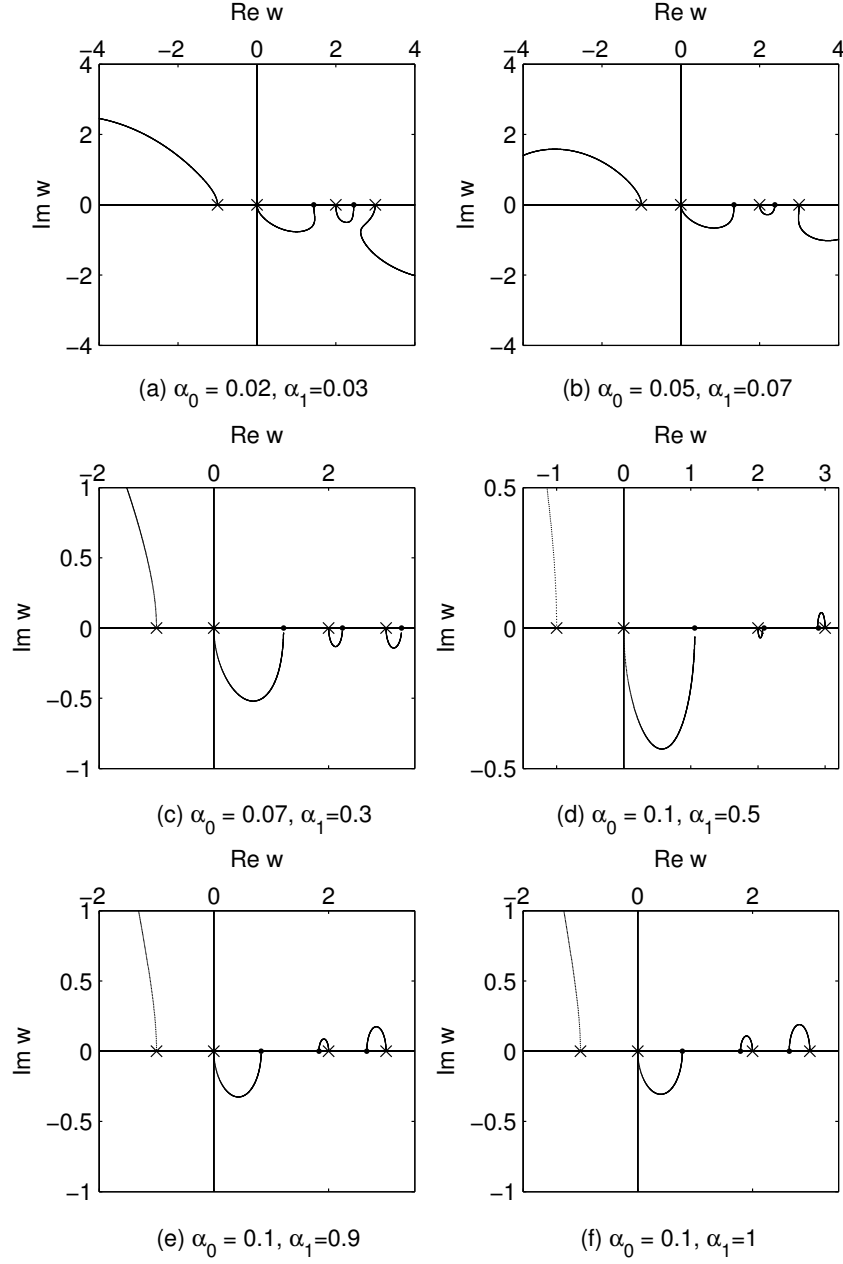


Figure 2.11: The longitudinal squared wave speeds of constrained anisotropic TRDTE theory. For each part, $\tilde{p}_1 = -1, \tilde{p}_2 = 2, \tilde{p}_3 = 3, \tilde{W}_1 = 1.5, \tilde{W}_2 = 2.5, \tilde{\sigma} = 1, \varepsilon = 1$.

2.4 Constrained isotropic TRDTE

2.4.1 Usual form of deformation-temperature constraints

The field equations for TRDTE of constrained anisotropic thermoelastic materials are (2.83) and (2.85) with the form of deformation-temperature constraint (2.81)

$$\left. \begin{aligned} \tilde{c}_{ijkl}u_{k,lj} - \beta_{ij}(\theta_{,j} + \alpha_1\dot{\theta}_{,j}) + \tilde{N}_{ij}\tilde{\eta}_{,j} &= \rho\ddot{u}_i, \\ T\beta_{qp}\dot{u}_{p,q} + \rho c(\dot{\theta} + \alpha_0\ddot{\theta}) + T\alpha\dot{\tilde{\eta}} - k_{ij}\theta_{,ij} &= 0, \\ \tilde{N}_{ij}u_{i,j} - \alpha\theta &= 0. \end{aligned} \right\} \quad (2.190)$$

For a material which is isotropic and incompressible at uniform temperature we use the following forms, see [27, (2.2)],

$$\tilde{c}_{ijkl} = \tilde{\lambda}\delta_{ij}\delta_{kl} + \tilde{\mu}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \quad \beta_{ij} = \beta\delta_{ij}, \quad k_{ij} = k\delta_{ij}, \quad \tilde{N}_{ij} = \tilde{N}\delta_{ij}. \quad (2.191)$$

With aid of (2.191) equations (2.190) may be written as

$$\left. \begin{aligned} (\tilde{\lambda} + \tilde{\mu})u_{j,ij} + \tilde{\mu}u_{i,jj} - \beta(\theta + \alpha_1\dot{\theta})_{,i} + \tilde{N}\tilde{\eta}_{,i} &= \rho\ddot{u}_i, \\ k\theta_{,ii} - T\beta\dot{u}_{j,j} - \alpha T\dot{\tilde{\eta}} &= \rho c(\dot{\theta} + \alpha_0\ddot{\theta}), \\ \tilde{N}u_{i,i} - \alpha\theta &= 0. \end{aligned} \right\} \quad (2.192)$$

The secular equation

Now we are looking for solutions of (2.192) in the form of plane harmonic waves (2.86). Similarly to Section 2.2.1, by substituting (2.86) into (2.192) to get the following system of algebraic equations

$$\left. \begin{aligned} [(\tilde{\mu} - \rho s^{-2})\delta_{ij} + (\tilde{\lambda} + \tilde{\mu})n_i n_j]U_j + i\beta(\omega s)^{-1}n_i(1 - i\omega\alpha_1)\Theta - i\tilde{N}n_i(\omega s)^{-1}\tilde{H} &= 0, \\ T\beta\omega s n_j U_j + (\omega s^2 k - i\rho c(1 - i\omega\alpha_0))\Theta - i\alpha T\tilde{H} &= 0, \\ \tilde{N}n_i i\omega s U_i - \alpha\Theta &= 0. \end{aligned} \right\} \quad (2.193)$$

Eliminate Θ and \tilde{H} between equations (2.193). Firstly, from (2.193)₃ we can write Θ as follows

$$\Theta = \frac{i\omega s \tilde{N} n_i U_i}{\alpha}. \quad (2.194)$$

Substituting (2.194) into (2.193)₂, we get

$$\tilde{H} = -i\alpha^{-1}\beta\omega sn_j U_j + \left(\frac{\omega s \tilde{N} n_i}{\alpha^2 T}\right)(\omega s^2 k - i\rho c(1 - i\omega\alpha_0))U_i. \quad (2.195)$$

Inserting (2.194) and (2.195) into (2.193)₁, we get

$$\begin{aligned} & \left[(\tilde{\mu} - w)\delta_{ij} + (\tilde{\lambda} + \tilde{\mu})n_i n_j \right] U_j + i\beta(\omega s)^{-1}n_i(1 - i\omega\alpha_1) \left(\frac{i\omega s \tilde{N} n_i U_i}{\alpha} \right) \\ & - i\tilde{N}(\omega s)^{-1}n_i \left[-i\alpha^{-1}\beta\omega sn_j U_j + \frac{\omega s \tilde{N} n_i}{\alpha^2 T}(\omega s^2 k - i\rho c(1 - i\omega\alpha_0))U_i \right] = 0. \end{aligned}$$

After simplifying and rearranging the equation we obtain

$$\begin{aligned} & \left\{ (\tilde{\mu} - w)\delta_{ij} + \left[(\tilde{\lambda} + \tilde{\mu} - \alpha^{-1}\beta\tilde{N}(1 + (1 - i\omega\alpha_1))) \right. \right. \\ & \quad \left. \left. - \left(\frac{\rho c \tilde{N}^2}{\alpha^2 T} \right) \left((1 - i\omega\alpha_0) + \frac{i\omega k}{wc} \right) \right] n_i n_j \right\} U_j = 0, \quad (2.196) \end{aligned}$$

which gives in direct notation the secular equation

$$\begin{aligned} & \det \left\{ (\tilde{\mu} - w)\mathbf{1} + \left[(\tilde{\lambda} + \tilde{\mu} - \alpha^{-1}\beta\tilde{N}(2 - i\omega\alpha_1)) \right. \right. \\ & \quad \left. \left. - \left(\frac{\rho c \tilde{N}^2}{\alpha^2 T} \right) \left((1 - i\omega\alpha_0) + \frac{i\omega k}{wc} \right) \right] \mathbf{n} \otimes \mathbf{n} \right\} = 0. \quad (2.197) \end{aligned}$$

Non-dimensionalize this equation by applying the dimensionless quantities (2.57) and further dimensionless quantities

$$\alpha' = \alpha T, \quad c' = \rho c T / \gamma, \quad \omega^* = \gamma c / k,$$

to get

$$\begin{aligned} & \det \left\{ (\tilde{\mu}' - w')\mathbf{1} + \left[(\tilde{\lambda}' + \tilde{\mu}' - \frac{(\varepsilon c')^{1/2} \tilde{N}}{\alpha'}(2 - i\omega' \alpha'_1)) \right. \right. \\ & \quad \left. \left. - \left(\frac{c' \tilde{N}^2}{\alpha'^2} \right) \left((1 - i\omega' \alpha'_0) - \frac{i\omega'}{w'} \right) \right] \mathbf{n} \otimes \mathbf{n} \right\} = 0. \quad (2.198) \end{aligned}$$

Now by using the standard identity (2.60), and dropping the dashes for convenience, we get the secular equation as follows

$$(w - \tilde{\mu})^2 \left[w^2 - w \left(1 + (\varepsilon c)^{1/2} \tilde{N} \alpha^{-1} (2 - i\omega \alpha_1) - c \tilde{N}^2 \alpha^{-2} (1 - i\omega \alpha_0) \right) + i\omega c \tilde{N}^2 \alpha^{-2} \right] = 0. \quad (2.199)$$

This is the secular equation for isotropic TRDTE which is constrained by the usual deformation temperature constraint and has not previously appeared in the literature.

The repeated root $w = \tilde{\mu}$ represents two purely elastic transverse waves, and longitudinal waves are represented by the two roots of the following quadratic equation

$$\alpha^2 w^2 - w \left(\alpha^2 - (\varepsilon c)^{1/2} \alpha \tilde{N} (2 - i\omega \alpha_1) - c \tilde{N}^2 (1 - i\omega \alpha_0) \right) + i\omega c \tilde{N}^2 = 0. \quad (2.200)$$

Equation (2.200) may be written as

$$w^2 - w \left(1 - \varepsilon^{1/2} \tilde{\sigma} (2 - i\omega \alpha_1) - \tilde{\sigma}^2 (1 - i\omega \alpha_0) \right) + i\omega \tilde{\sigma}^2 = 0, \quad (2.201)$$

where

$$\tilde{\sigma} = \frac{c^{1/2} \tilde{N}}{\alpha}, \quad (2.202)$$

which satisfies $0 < \tilde{\sigma} < \infty$ because $\tilde{N} \neq 0$ and $\alpha \neq 0$. The ratio $\tilde{\sigma}$ is a measure of the relative importance of the mechanical and thermal parts of the constraint (2.193)₃ because $\tilde{\sigma} \rightarrow 0$ represents a purely isothermal constraint and $\tilde{\sigma} \rightarrow \infty$ represents a purely mechanical constraint. Equation (2.201) is the final form which represents the squared wave speeds of purely longitudinal waves propagating in an isotropic thermoelastic material that is incompressible at uniform temperature. In this equation there are two parameters ε and $\tilde{\sigma}$ which affect the behaviour of the roots of this equation. Now we want to examine stability in the special cases $\tilde{\sigma} = 0$ and $\tilde{\sigma} \rightarrow \infty$.

Case 1: The isothermal constraint ($\tilde{N} = 0, \alpha \neq 0$)

Putting $\tilde{N} = 0$ into equation (2.200) gives the quadratic equation with roots

$$w_1 = 0, \quad w_2 = 1 \quad (2.203)$$

in which (2.203)₁ is a spurious root and (2.203)₂ is the elastic equation. Both roots satisfy $\text{Im } w(\omega) \leq 0$ and are therefore stable. With $\tilde{N} = 0$, (2.193)₃ implies $\theta = 0$, so that (2.193)₁ gives the equation for isotropic isothermal elasticity and (2.193)₂ solves for $\tilde{\eta}$.

Case 2: The purely mechanical constraint ($\tilde{N} \neq 0, \alpha = 0$)

Inserting $\alpha = 0$ into equation (2.200) reduces it to the single branch

$$w = \frac{-i\omega}{1 - i\omega \alpha_0}. \quad (2.204)$$

Equation (2.204) is purely diffusive for small ω and, although a purely mechanical constraint is not physically realistic, also satisfies the stability criterion $\text{Im } w(\omega) \leq 0$. In this case putting $\alpha = 0$ into (2.193)₃ leads to absolute incompressibility, a purely mechanical constraint where no volume changes are possible.

In the general case, in which neither \tilde{N} nor α is equal to zero, it is convenient to go back to equation (2.201). The roots of (2.201) are

$$w_{1,2} = \bar{A} \pm \left[\bar{A}^2 - i\omega\tilde{\sigma}^2 \right]^{\frac{1}{2}}, \quad (2.205)$$

where

$$\begin{aligned} \bar{A} &= \frac{1}{2} \left[1 - \varepsilon^{1/2} \tilde{\sigma} (2 - i\omega\alpha_1) - \tilde{\sigma}^2 (1 - i\omega\alpha_0) \right], \\ &= \frac{1}{2} \left[1 + \varepsilon - (\varepsilon^{1/2} + \tilde{\sigma})^2 + i\omega\tilde{\sigma}(\alpha_0\tilde{\sigma} + \alpha_1\varepsilon^{1/2}) \right]. \end{aligned} \quad (2.206)$$

For fixed $\varepsilon \geq 0$, as $\tilde{\sigma}$ increases from 0 to ∞ , $\text{Re } \bar{A}$ decreases from $\frac{1}{2}$ to $-\infty$. $\text{Re } \bar{A}$ becomes 0 for a critical value of $\tilde{\sigma}$ given by

$$\tilde{\sigma}_c = (1 + \varepsilon)^{1/2} - \varepsilon^{1/2}. \quad (2.207)$$

In the special case where $\tilde{\sigma} = \tilde{\sigma}_c$, so $\text{Re } \bar{A} = 0$ in (2.205), we get

$$\begin{aligned} w &= \pm (-i\omega\tilde{\sigma}_c^2)^{\frac{1}{2}} + O(\omega), \\ &= \pm (e^{-i\frac{\pi}{2}} \tilde{\sigma}_c^2 \omega)^{\frac{1}{2}} + O(\omega), \end{aligned}$$

So,

$$w = \pm e^{-i\frac{\pi}{4}} \omega^{\frac{1}{2}} \tilde{\sigma}_c + O(\omega). \quad (2.208)$$

This equation (2.208) is similar to its counterpart in the classical thermoelasticity, see [29, (3.19)].

Low frequency expansions

When $\omega = 0$, the roots of the secular equation (2.201) are

$$w_1 = 1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2, \quad w_2 = 0. \quad (2.208a)$$

Now we are looking for the roots of the secular equation (2.201) as $\omega \rightarrow 0$.

The first root is

$$w_1 = 1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2 + A(i\omega) + O(\omega^2), \quad (2.208b)$$

inserting (2.208b) into the secular equation (2.201) to find A , we get

$$A = \tilde{\sigma}(\alpha_1\varepsilon^{1/2} + \alpha_0\tilde{\sigma}) - \frac{\tilde{\sigma}^2}{1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2}.$$

Then,

$$w_1 = 1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2 + i\omega\tilde{\sigma}\left[(\alpha_1\varepsilon^{1/2} + \alpha_0\tilde{\sigma}) - \frac{\tilde{\sigma}}{1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2}\right] + O(\omega^2).$$

The second roots may be written as

$$w_2 = B(i\omega) + O(\omega^2), \quad (2.208c)$$

inserting (2.202c) into the secular equation (2.201) to find B , we obtain

$$B = \frac{\tilde{\sigma}^2}{1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2}.$$

Thus,

$$w_2 = \frac{i\omega\tilde{\sigma}^2}{1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2} + O(\omega^2).$$

If $\tilde{\sigma} > \tilde{\sigma}_c$ we find that $\text{Im } w_1(\omega) > 0$ then w_1 is unstable, and $\text{Im } w_2(\omega) < 0$, so w_2 is stable, but if $\tilde{\sigma} < \tilde{\sigma}_c$ we cannot tell the sign of $\text{Im } w_1(\omega)$ because it depends on the relative values of the quantities occurring, but it is clear $\text{Im } w_2(\omega) > 0$, thus $w_2(\omega)$ is unstable. If $\tilde{\sigma} = \tilde{\sigma}_c$ the analysis is not valid because the denominator will be zero, so the roots of the secular equation in this case are (2.208).

High frequency expansions

In the high frequency as $\omega \rightarrow \infty$, so $\frac{1}{\omega} \rightarrow 0$. The secular equation (2.201) may be written as

$$\left(\frac{1}{i\omega}\right)w^2 - w\left\{\left(\frac{1}{i\omega}\right) - \varepsilon^{1/2}\tilde{\sigma}\left[\left(\frac{2}{i\omega}\right) - \alpha_1\right] - \tilde{\sigma}^2\left[\left(\frac{1}{i\omega}\right) - \alpha_0\right]\right\} + \tilde{\sigma}^2 = 0. \quad (2.208d)$$

The secular equation (2.208d) at $\frac{1}{i\omega} = 0$ becomes

$$-w\tilde{\sigma}(\varepsilon^{1/2}\alpha_1 + \tilde{\sigma}\alpha_0) + \tilde{\sigma}^2 = 0.$$

The roots now are

$$w_1 = \frac{\tilde{\sigma}}{\varepsilon^{1/2}\alpha_1 + \tilde{\sigma}\alpha_0}, \quad w_2 \rightarrow \infty.$$

Now we are looking at the roots as $\frac{1}{\omega} \rightarrow 0$. The roots in this case may be written as

$$w_1 = \frac{\tilde{\sigma}}{\varepsilon^{1/2}\alpha_1 + \tilde{\sigma}\alpha_0} + A(i\omega)^{-1} + O(\omega^{-2}), \quad (2.208e)$$

and

$$w_2 = B(i\omega) + C + O(\omega^{-1}). \quad (2.208f)$$

Firstly, inserting (2.208e) into the secular equation (2.208d) to find A we get

$$A = \frac{1}{(\varepsilon^{1/2}\alpha_1 + \tilde{\sigma}\alpha_0)^2} \left\{ \frac{\tilde{\sigma}}{\varepsilon^{1/2}\alpha_1 + \tilde{\sigma}\alpha_0} - (1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2) \right\}.$$

So, the first root becomes

$$w_1 = \frac{\tilde{\sigma}}{\varepsilon^{1/2}\alpha_1 + \tilde{\sigma}\alpha_0} - \frac{i\omega^{-1}}{(\varepsilon^{1/2}\alpha_1 + \tilde{\sigma}\alpha_0)^2} \left[\frac{\tilde{\sigma}}{\varepsilon^{1/2}\alpha_1 + \tilde{\sigma}\alpha_0} - (1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2) \right] + O(\omega^{-2}).$$

If $\tilde{\sigma} < \tilde{\sigma}_c$, we cannot tell about the sign of $\text{Im } w_1(\omega)$, and if $\tilde{\sigma} > \tilde{\sigma}_c$ we find that $\text{Im } w_1(\omega) < 0$ so w_1 is stable. But if $\tilde{\sigma} = \tilde{\sigma}_c$, the analysis is not valid and we can get the roots of this special case as (2.208).

Similarly, substituting (2.208f) into (2.208d) to find B and C we get

$$B = \tilde{\sigma}(\varepsilon^{1/2}\alpha_1 + \tilde{\sigma}\alpha_0),$$

and

$$C = (1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2) - \frac{\tilde{\sigma}}{\varepsilon^{1/2}\alpha_1 + \tilde{\sigma}\alpha_0}.$$

Thus, the second root may be written as

$$w_2 = i\omega\tilde{\sigma}(\varepsilon^{1/2}\alpha_1 + \tilde{\sigma}\alpha_0) + (1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2) - \frac{\tilde{\sigma}}{\varepsilon^{1/2}\alpha_1 + \tilde{\sigma}\alpha_0} + O(\omega^{-2}).$$

It is clear that $\text{Im } w_2(\omega) > 0$, thus w_2 is unstable in the high frequency limit.

Now we consider the two former special cases.

Case 1: The isothermal constraint viewed as the limit $\tilde{\sigma} \rightarrow 0$

From (2.205)

$$w_{1,2} = \frac{1}{2} \left(1 - \varepsilon^{1/2}\tilde{\sigma}(2 - i\omega\alpha_1) - \tilde{\sigma}^2(1 - i\omega\alpha_0) \right) \pm \frac{1}{2} \left\{ \left(1 - \varepsilon^{1/2}\tilde{\sigma}(2 - i\omega\alpha_1) - \tilde{\sigma}^2(1 - i\omega\alpha_0) \right)^2 - 4i\omega\tilde{\sigma}^2 \right\}^{1/2}.$$

After expanding and using the binomial expansion we obtain the first root

$$w_1 = 1 - \varepsilon^{1/2}\tilde{\sigma}(2 - i\omega\alpha_1) - \tilde{\sigma}^2(1 - i\omega\alpha_0) - i\omega\tilde{\sigma}^2 + O(\tilde{\sigma}^3). \quad (2.209)$$

Rewrite (2.209) as

$$w_1 = 1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2 + i\omega\tilde{\sigma}(\alpha_1\varepsilon^{1/2} + \alpha_0\tilde{\sigma} - \tilde{\sigma}) + O(\tilde{\sigma}^3), \quad (2.209a)$$

and the second root as

$$w_2 = i\omega\tilde{\sigma}^2 + O(\tilde{\sigma}^3). \quad (2.210)$$

Equation (2.209a) represents an unstable branch, for $\tilde{\sigma}$ small enough, starting from the point (putting $\omega = 0$ in equation (2.209))

$$w = 1 - 2\varepsilon^{\frac{1}{2}}\tilde{\sigma} - \tilde{\sigma}^2,$$

and equation (2.210) describes an unstable branch starting from the origin. Also equation (2.209) is similar to its counterpart in classical thermoelasticity [29, (3.21)].

Case 2: The purely mechanical constraint viewed as the limit $\tilde{\sigma} \rightarrow \infty$

When $\tilde{\sigma} \rightarrow \infty$, meaning $\frac{1}{\tilde{\sigma}}$ is small, from (2.205) after expanding and using the binomial expansion, we obtain

$$w_1 = \frac{-i\omega}{(1 - i\omega\alpha_0)} \left[1 - \varepsilon^{1/2}\tilde{\sigma}^{-1} \frac{(2 - i\omega\alpha_1)}{(1 - i\omega\alpha_0)} \right] + O(\tilde{\sigma}^{-2}). \quad (2.211)$$

$$w_2 = 1 - \varepsilon^{1/2}\tilde{\sigma}(2 - i\omega\alpha_1) - \tilde{\sigma}^2(1 - i\omega\alpha_0) + \frac{i\omega}{(1 - i\omega\alpha_0)} \left[1 - \varepsilon^{1/2}\tilde{\sigma}^{-1} \frac{(2 - i\omega\alpha_1)}{(1 - i\omega\alpha_0)} \right] + O(\tilde{\sigma}^{-2}). \quad (2.212)$$

Putting $\alpha_0 = \alpha_1 = 0$ in equations (2.211) and (2.212) we will return to classical thermoelasticity [29, (3.22)].

Numerical results

In Figure 2.12 we use $\varepsilon = 1, \alpha_0 = 0.1$ and $\alpha_1 = 0.2$, and w is plotted for a range of values of $\tilde{\sigma}$. We have two longitudinal waves one is finite and the other tends to infinity. There is \times at zero and $1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2$ marking the low frequency limits, and \circ indicating the high frequency limits. If $\tilde{\sigma} < \tilde{\sigma}_c$ both branches are unstable one of them starting from the origin and the other starting from the point $1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2$, see subfigures (a)–(c). If $\tilde{\sigma} > \tilde{\sigma}_c$ one branch only maintains the instability and starting

from $1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2$, and the other becomes stable and emanating from the origin, see subfigures (e) and (f). In the special case when $\tilde{\sigma} = \tilde{\sigma}_c$ the branches become a connected line passing through the origin at angle $-\pi/4$ to the real axis, see the subfigure (d).

Varying the parameters α_1 , α_0 and ε while changing the magnitude of ω does not have any substantive influence on the stability.

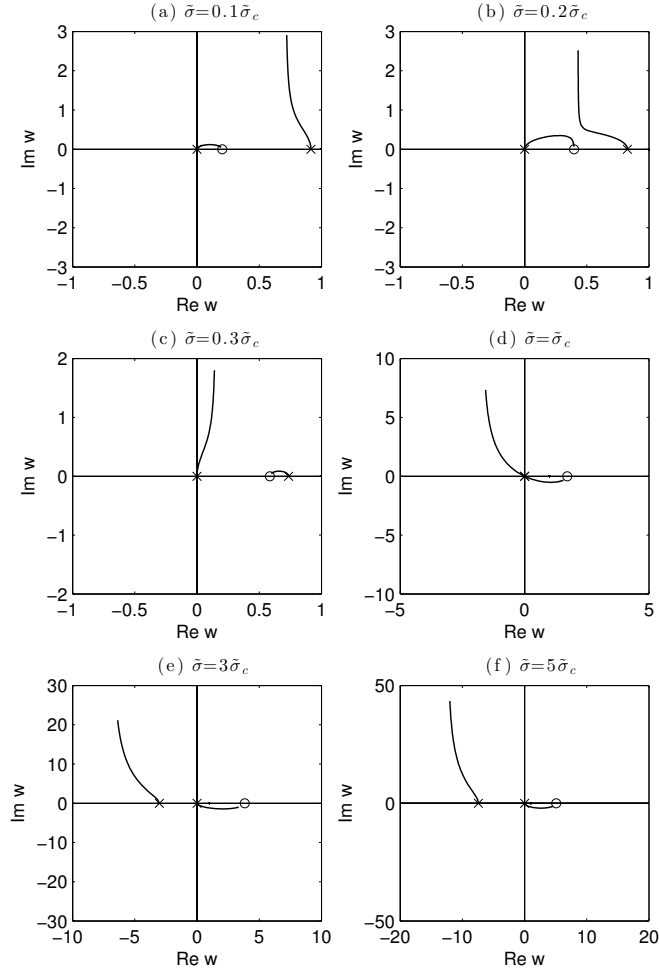


Figure 2.12: The longitudinal squared wave speeds of isotropic thermelastic material for Temperature-rate-dependent thermoelasticity theory with incompressibility at uniform temperature. For each part ($\varepsilon = 1, \alpha_0 = 0.1, \alpha_1 = 0.2$), (a) $\tilde{\sigma} = 0.1\tilde{\sigma}_c$, (b) $\tilde{\sigma} = 0.2\tilde{\sigma}_c$, (c) $\tilde{\sigma} = 0.3\tilde{\sigma}_c$, (d) $\tilde{\sigma} = \tilde{\sigma}_c$, (e) $\tilde{\sigma} = 3\tilde{\sigma}_c$, (f) $\tilde{\sigma} = 5\tilde{\sigma}_c$.

2.4.2 Alternative form of deformation-temperature constraints

In this section we will use equations (2.192)_{1,2} and the alternative form of deformation temperature constraint (2.144), to get the field equations for the alternative deformation-temperature constraint TRDTE theory for the isotropic case:

$$\left. \begin{aligned} (\tilde{\lambda} + \tilde{\mu})u_{j,ij} + \tilde{\mu}u_{i,jj} - \beta(\theta + \alpha_1\dot{\theta})_{,i} + \tilde{N}\tilde{\eta}_{,i} &= \rho\ddot{u}_i, \\ k\theta_{,ii} - T\beta\dot{u}_{j,j} - \alpha T\dot{\tilde{\eta}} &= \rho c(\dot{\theta} + \alpha_0\ddot{\theta}), \\ \tilde{N}u_{i,i} - \alpha(\theta + \alpha_0\dot{\theta}) &= 0. \end{aligned} \right\} \quad (2.213)$$

The secular equation

Now we are looking for solutions in the form of plane harmonic waves (2.86), by inserting (2.86) into (2.213) to get the system of algebraic equations

$$\left. \begin{aligned} [(\tilde{\mu} - \rho s^{-2})\delta_{ij} + (\tilde{\lambda} + \tilde{\mu})n_i n_j]U_j + i\beta(\omega s)^{-1}n_i(1 - i\omega\alpha_1)\Theta - i\tilde{N}n_i(\omega s)^{-1}\tilde{H} &= 0, \\ T\beta\omega s n_j U_j + (\omega s^2 k - i\rho c(1 - i\omega\alpha_0))\Theta - i\alpha T\tilde{H} &= 0, \\ \tilde{N}n_i i\omega s U_i - \alpha(1 - i\omega\alpha_0)\Theta &= 0. \end{aligned} \right\} \quad (2.214)$$

Eliminate Θ and \tilde{H} between (2.214), similarly to the previous sections. From (2.214)₃ we can rewrite Θ as follows

$$\Theta = \frac{i\omega s \tilde{N} n_i U_i}{\alpha(1 - i\omega\alpha_0)}. \quad (2.215)$$

Substituting (2.215) into (2.214)₂, we get

$$\tilde{H} = -i\alpha^{-1}\beta\omega s n_j U_j + \left(\frac{\omega s \tilde{N} n_j}{\alpha^2 T(1 - i\omega\alpha_0)} \right) (\omega s^2 k - i\rho c(1 - i\omega\alpha_0)) U_i. \quad (2.216)$$

Inserting (2.215) and (2.216) into (2.214)₁, we get

$$\begin{aligned} & \left[(\tilde{\mu} - w)\delta_{ij} + (\tilde{\lambda} + \tilde{\mu})n_i n_j \right] U_j + i\beta(\omega s)^{-1}n_i(1 - i\omega\alpha_1) \left(\frac{i\omega s \tilde{N} n_i U_i}{\alpha(1 - i\omega\alpha_0)} \right) - \\ & i\tilde{N}(\omega s)^{-1}n_j \left[-i\alpha^{-1}\beta\omega s n_j U_j + \frac{\omega s \tilde{N} n_i}{\alpha^2 T(1 - i\omega\alpha_0)} (\omega s^2 k - i\rho c(1 - i\omega\alpha_0)) U_i \right] = 0. \end{aligned} \quad (2.217)$$

After simplifying and rearranging the equation we obtain

$$\begin{aligned} & \left\{ (\tilde{\mu} - w)\delta_{ij} + \left[\tilde{\lambda} + \tilde{\mu} - \alpha^{-1}\beta\tilde{N} \left(1 + \frac{1 - i\omega\alpha_1}{1 - i\omega\alpha_0} \right) - \right. \right. \\ & \left. \left. \left(\frac{\rho c \tilde{N}^2}{\alpha^2 T(1 - i\omega\alpha_0)} \right) \left((1 - i\omega\alpha_0) + \frac{i\omega k}{wc} \right) \right] n_i n_j \right\} U_j = 0, \end{aligned} \quad (2.218)$$

which gives in direct notation the secular equation

$$\det \left\{ (\tilde{\mu} - w)\mathbf{1} + \left[\tilde{\lambda} + \tilde{\mu} - \alpha^{-1}\beta\tilde{N} \left(1 + \frac{1 - i\omega\alpha_1}{1 - i\omega\alpha_0} \right) - \left(\frac{\rho c \tilde{N}^2}{\alpha^2 T (1 - i\omega\alpha_0)} \right) \left((1 - i\omega\alpha_0) + \frac{i\omega k}{wc} \right) \right] \mathbf{n} \otimes \mathbf{n} \right\} = 0. \quad (2.219)$$

Non-dimensionalize this equation by applying the dimensionless quantities (2.57) to get

$$\det \left\{ (\tilde{\mu}' - w')\mathbf{1} + \left[\left(\tilde{\lambda}' + \tilde{\mu}' - \frac{(\varepsilon c')^{1/2}\tilde{N}}{\alpha'} \left(1 + \frac{1 - i\omega'\alpha'_1}{1 - i\omega'\alpha'_0} \right) \right) - \left(\frac{c'\tilde{N}^2}{\alpha'^2 (1 - i\omega'\alpha'_0)} \right) \left((1 - i\omega'\alpha'_0) + \frac{i\omega'}{w'} \right) \right] \mathbf{n} \otimes \mathbf{n} \right\} = 0. \quad (2.220)$$

Now by using the standard identity (2.60), dropping the dashes for convenience, we get the secular equation as follows

$$(w - \tilde{\mu})^2 \left[\alpha^2 w^2 - w \left(\alpha^2 - (\varepsilon c)^{1/2} \alpha \tilde{N} \left(1 + \frac{1 - i\omega\alpha_1}{1 - i\omega\alpha_0} \right) - c \tilde{N}^2 \right) + \frac{i\omega c \tilde{N}^2}{1 - i\omega\alpha_0} \right] = 0. \quad (2.221)$$

This is the secular equation for isotropic TRDTE which is constrained by the alternative deformation temperature constraint and has not previously appeared in the literature.

The repeated root $w = \tilde{\mu}$ represents two purely elastic transverse waves, and longitudinal waves are represented by the roots of the following quadratic equation

$$\alpha^2 w^2 - w \left(\alpha^2 - (\varepsilon c)^{1/2} \alpha \tilde{N} \left(1 + \frac{1 - i\omega\alpha_1}{1 - i\omega\alpha_0} \right) - c \tilde{N}^2 \right) + \frac{i\omega c \tilde{N}^2}{1 - i\omega\alpha_0} = 0. \quad (2.222)$$

By dividing by α^2 we get

$$w^2 - w \left(1 - \frac{(\varepsilon c)^{1/2} \tilde{N}}{\alpha} \left(1 + \frac{1 - i\omega\alpha_1}{1 - i\omega\alpha_0} \right) - \frac{c \tilde{N}^2}{\alpha^2} \right) + \frac{i\omega c \tilde{N}^2}{\alpha^2 (1 - i\omega\alpha_0)} = 0. \quad (2.223)$$

We can rewrite equation (2.223) as

$$w^2 + w \left(\tilde{\sigma}^2 + \varepsilon^{1/2} \tilde{\sigma} \left(1 + \frac{1 - i\omega\alpha_1}{1 - i\omega\alpha_0} \right) - 1 \right) + \frac{i\omega \tilde{\sigma}^2}{(1 - i\omega\alpha_0)} = 0, \quad (2.224)$$

This equation may be written as

$$w^2 (1 - i\omega\alpha_0) - w \left[(1 - i\omega\alpha_0) - \varepsilon^{1/2} \tilde{\sigma} \left(2 - i\omega(\alpha_1 + \alpha_0) \right) - \tilde{\sigma}^2 (1 - i\omega\alpha_0) \right] + i\omega \tilde{\sigma}^2 = 0. \quad (2.224a)$$

where $\tilde{\sigma}$ is defined earlier in (2.202). Returning to the special cases that were discussed in Section 2.4.1 ($\tilde{N} = 0, \alpha \neq 0$) and ($\tilde{N} \neq 0, \alpha = 0$), we will get the same equations (2.203) and (2.204) respectively. In examining the more general case in which neither \tilde{N} nor α is equal to zero, it is convenient to go back to equation (2.224). The roots of (2.224) are

$$w_{1,2} = \bar{A} \pm \left[\bar{A}^2 - \frac{i\omega\tilde{\sigma}^2}{1 - i\omega\alpha_0} \right]^{\frac{1}{2}}, \quad (2.225)$$

where

$$\bar{A} = \frac{1}{2} \left[1 - \varepsilon^{1/2} \tilde{\sigma} \left(1 + \frac{1 - i\omega\alpha_1}{1 - i\omega\alpha_0} \right) - \tilde{\sigma}^2 \right]. \quad (2.226)$$

Similarly to Section 2.4.1 for fixed $\varepsilon \geq 0$ and fixed $\omega = 0$, as $\tilde{\sigma}$ increases from 0 to ∞ , $\text{Re } \bar{A}$ decreases from $\frac{1}{2}$ to $-\infty$,

$$\text{Re } \bar{A} = \frac{1}{2} \left[1 - \varepsilon^{1/2} \tilde{\sigma} \left(1 + \frac{1 + \omega^2\alpha_0\alpha_1}{1 + \omega^2\alpha_0^2} \right) - \tilde{\sigma}^2 \right].$$

Putting $\omega = 0$, we get

$$\text{Re } \bar{A} = \frac{1}{2} \left[1 - 2\varepsilon^{1/2} \tilde{\sigma} - \tilde{\sigma}^2 \right].$$

$\text{Re } \bar{A}$ becomes 0 at $\omega = 0$ for a critical value of $\tilde{\sigma}$ given by

$$\tilde{\sigma}_c = (1 + \varepsilon)^{1/2} - \varepsilon^{1/2}. \quad (2.227)$$

In the special case where $\tilde{\sigma} = \tilde{\sigma}_c$, (2.225) gives

$$\begin{aligned} w &= \pm(-i\omega\tilde{\sigma}_c^2)^{\frac{1}{2}} + O(\omega), \\ &= \pm(e^{-i\frac{\pi}{2}}\tilde{\sigma}_c^2\omega)^{\frac{1}{2}} + O(\omega), \end{aligned}$$

So,

$$w = \pm e^{-i\frac{\pi}{4}} \omega^{\frac{1}{2}} \tilde{\sigma}_c + O(\omega). \quad (2.228)$$

Equation (2.228) is similar to its counterpart (2.208) in Section 2.4.1.

Low frequency expansions

In the low frequency at $\omega = 0$ the secular equation (2.224) becomes

$$w^2 - w(1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2) = 0,$$

so the roots of this equation are

$$w_1 = 1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2, \quad w_2 = 0.$$

Now the roots of the secular equation (2.224) when $\omega \rightarrow 0$ may be written as

$$w_1 = 1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2 + A(i\omega) + O(\omega^2). \quad (2.228a)$$

$$w_2 = B(i\omega) + C + O(\omega^2). \quad (2.228b)$$

In order to get A we need to insert (2.228a) into (2.224), so we get

$$A = \tilde{\sigma} \left[\varepsilon^{1/2}(\alpha_1 - \alpha_0) - \frac{\tilde{\sigma}}{1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2} \right].$$

Thus, the first root is

$$w_1 = 1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2 + i\omega\tilde{\sigma} \left[\varepsilon^{1/2}(\alpha_1 - \alpha_0) - \frac{\tilde{\sigma}}{1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2} \right] + O(\omega^2).$$

To get the second root we need firstly to find B , by substituting (2.228b) into (2.224), so

$$B = \frac{\tilde{\sigma}^2}{1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2},$$

thus,

$$w_2 = \frac{i\omega\tilde{\sigma}^2}{1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2} + O(\omega^2).$$

It is clear that if $\tilde{\sigma} > \tilde{\sigma}_c$ we find that w_1 is unstable and w_2 is stable, if $\tilde{\sigma} < \tilde{\sigma}_c$ we cannot tell about the sign of $\text{Im } w_1(\omega)$ because it depends on the relative values of the quantities occurring, while $\text{Im } w_2(\omega) > 0$ thus w_2 is unstable. But if $\tilde{\sigma} = \tilde{\sigma}_c$ the analysis is not valid and we can get the roots of the secular equation in the special case as (2.228).

High frequency expansions

The secular equation (2.224a) after dividing by $i\omega$ may be written as

$$w^2 \left(\frac{1}{i\omega} - \alpha_0 \right) - w \left\{ \left(\frac{1}{i\omega} - \alpha_0 \right) - \varepsilon^{1/2}\tilde{\sigma} \left[\frac{2}{i\omega} - (\alpha_1 + \alpha_0) \right] - \frac{\tilde{\sigma}^2}{i\omega} + \alpha_0\tilde{\sigma}^2 \right\} + \tilde{\sigma}^2 = 0. \quad (2.228c)$$

In the high frequency limits as $\omega \rightarrow \infty$ we find that $\frac{1}{\omega} \rightarrow 0$, so the secular equation (2.228c) becomes

$$\alpha_0 w^2 - w \left[\alpha_0 - \varepsilon^{1/2}\tilde{\sigma}(\alpha_1 + \alpha_0) - \alpha_0\tilde{\sigma}^2 \right] - \tilde{\sigma}^2 = 0.$$

Define

$$H(w) \equiv \alpha_0 w^2 - w \left[\alpha_0 - \varepsilon^{1/2} \tilde{\sigma} (\alpha_1 + \alpha_0) - \alpha_0 \tilde{\sigma}^2 \right] - \tilde{\sigma}^2.$$

To determine the positions of zeros of $H(w)$ we need to examine the sign changes

$$\begin{aligned} H(-\infty) &= \infty > 0, \\ H(0) &= -\tilde{\sigma}^2 < 0, \\ H(\infty) &= \infty > 0. \end{aligned} \tag{2.228d}$$

So we have in the high frequency limit, real roots \bar{h}_1 and \bar{h}_2 of $H(w) = 0$ satisfying

$$-\infty < \bar{h}_1 < 0 < \bar{h}_2 < \infty. \tag{2.228e}$$

They satisfy the following quadratic polynomial

$$\bar{h}(w) = (w - \bar{h}_1)(w - \bar{h}_2), \tag{2.228f}$$

and we must have

$$H(w) = \alpha_0 \bar{h}(w).$$

Collect terms in $i\omega$ in (2.224a) together:

$$w^2 - w(1 - 2\varepsilon^{1/2} \tilde{\sigma} - \tilde{\sigma}^2) - i\omega \left[\alpha_0 w^2 - w(\alpha_0 - \varepsilon^{1/2} \tilde{\sigma} (\alpha_1 + \alpha_0) - \alpha_0 \tilde{\sigma}^2) - \tilde{\sigma}^2 \right] = 0. \tag{2.228g}$$

With aid of (2.228f) equation (2.228g) may be written as

$$w^2 - w(1 - 2\varepsilon^{1/2} \tilde{\sigma} - \tilde{\sigma}^2) - i\omega \alpha_0 (w - \bar{h}_1)(w - \bar{h}_2) = 0. \tag{2.228h}$$

When $\omega \rightarrow \infty$ the roots written as

$$w_1 = \bar{h}_1 + A(i\omega)^{-1} + O(\omega^{-2}), \quad w_2 = \bar{h}_2 + B(i\omega)^{-1} + O(\omega^{-2}). \tag{2.228i}$$

To get A and B we need to insert (2.228i) into (2.228h), we find that

$$A = \frac{\bar{h}_1(\bar{h}_1 - (1 - 2\varepsilon^{1/2} \tilde{\sigma} - \tilde{\sigma}^2))}{\alpha_0(\bar{h}_1 - \bar{h}_2)},$$

and

$$B = \frac{\bar{h}_2(\bar{h}_2 - (1 - 2\varepsilon^{1/2} \tilde{\sigma} - \tilde{\sigma}^2))}{\alpha_0(\bar{h}_2 - \bar{h}_1)}.$$

Thus, the roots are

$$w_1 = \bar{h}_1 - i\omega^{-1} \left\{ \frac{\bar{h}_1(\bar{h}_1 - (1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2))}{\alpha_0(\bar{h}_1 - \bar{h}_2)} \right\} + O(\omega^{-2}),$$

and

$$w_2 = \bar{h}_2 - i\omega^{-1} \left\{ \frac{\bar{h}_2(\bar{h}_2 - (1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2))}{\alpha_0(\bar{h}_2 - \bar{h}_1)} \right\} + O(\omega^{-2}).$$

If $\tilde{\sigma} < \tilde{\sigma}_c$, we find that $\text{Im } w_1(\omega) > 0$ thus w_1 is unstable but we cannot tell about the sign of $\text{Im } w_2(\omega)$ because it depends on the sign of the relative values of the quantities occurring. If $\tilde{\sigma} > \tilde{\sigma}_c$ we cannot tell about the sign of $\text{Im } w_1(\omega)$ but $\text{Im } w_2(\omega) < 0$ thus w_2 is stable. If $\tilde{\sigma} = \tilde{\sigma}_c$, the analysis is not valid and we get the roots of the secular equation in the special case as (2.228).

Now we consider the two former special cases.

Case 1: The isothermal constraint viewed as the limit $\tilde{\sigma} \rightarrow 0$

Equation (2.225) may be written as

$$w_{1,2} = \frac{1}{2} \left(1 - \varepsilon^{1/2}\tilde{\sigma} \left(1 + \frac{1 - i\omega\alpha_1}{1 - i\omega\alpha_0} \right) - \tilde{\sigma}^2 \right) \pm \frac{1}{2} \left\{ \left(1 - \varepsilon^{1/2}\tilde{\sigma} \left(1 + \frac{1 - i\omega\alpha_1}{1 - i\omega\alpha_0} \right) - \tilde{\sigma}^2 \right)^2 - \frac{4i\omega\tilde{\sigma}^2}{1 - i\omega\alpha_0} \right\}^{1/2}.$$

After expanding and using the binomial expansion we get

$$w_1 = 1 - \varepsilon^{1/2}\tilde{\sigma} \left(1 + \frac{1 - i\omega\alpha_1}{1 - i\omega\alpha_0} \right) - \tilde{\sigma}^2 - \frac{i\omega\tilde{\sigma}^2}{1 - i\omega\alpha_0} + O(\tilde{\sigma}^3), \quad (2.229)$$

$$w_2 = \frac{i\omega\tilde{\sigma}^2}{1 - i\omega\alpha_0} + O(\omega^3). \quad (2.230)$$

Equation (2.229) represents a stable branch starting from the point (putting $\omega = 0$ in equation (2.229))

$$w = 1 - \varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2,$$

and equation (2.230) describes an unstable branch starting from the origin.

Case 2: The purely mechanical constraint viewed as the limit $\tilde{\sigma} \rightarrow \infty$

When $\tilde{\sigma} \rightarrow \infty$ means $\frac{1}{\tilde{\sigma}}$ is small, from (2.224), after expanding and using the binomial expansion, we obtain

$$w_1 = \frac{-i\omega}{1 - i\omega\alpha_0} \left(1 - \varepsilon^{1/2}\tilde{\sigma}^{-1} \left(1 + \frac{1 - i\omega\alpha_1}{1 - i\omega\alpha_0} \right) \right) + O(\tilde{\sigma}^{-2}). \quad (2.231)$$

$$w_2 = 1 - \varepsilon^{1/2} \tilde{\sigma} \left(1 + \frac{1 - i\omega\alpha_1}{1 - i\omega\alpha_0} \right) - \tilde{\sigma}^2 + \frac{i\omega}{1 - i\omega\alpha_0} \left(1 - \varepsilon^{1/2} \tilde{\sigma}^{-1} \left(1 + \frac{1 - i\omega\alpha_1}{1 - i\omega\alpha_0} \right) \right) + O(\tilde{\sigma}^{-2}). \quad (2.232)$$

Putting $\alpha_0 = \alpha_1 = 0$ in equations (2.231) and (2.232) we will recover the roots of the secular equation in the low and high frequencies in the classical thermoelasticity, see [29, (3.21)–(3.22)].

Numerical results

In each of Figures 2.14 and 2.15 we use $\alpha_0 = 0.01$ and $\alpha_1 = 0.02$ and we choose $\varepsilon = 0$ in Figure 2.14 and $\varepsilon = 1$ in Figure 2.15. In both of them we have two longitudinal waves one stable and the other unstable and both finite. The low frequency limits are indicated by a \times and the high frequency limits are indicated by a \circ . If $\tilde{\sigma} < \tilde{\sigma}_c$ the stable branch starting from the point $1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2$, and the unstable branch beginning at the origin, see sub-figures (a) and (b). But this situation is reversed if $\tilde{\sigma} > \tilde{\sigma}_c$ as shown in sub-figures (d)–(f). In the special case when $\tilde{\sigma} = \tilde{\sigma}_c$ the branches become a connected line passing through the origin at angle $-\pi/4$ to the real axis, see sub-figure (c).

Varying the parameters α_1, α_0 while changing the magnitude of ω does not have any substantive influence on the stability.

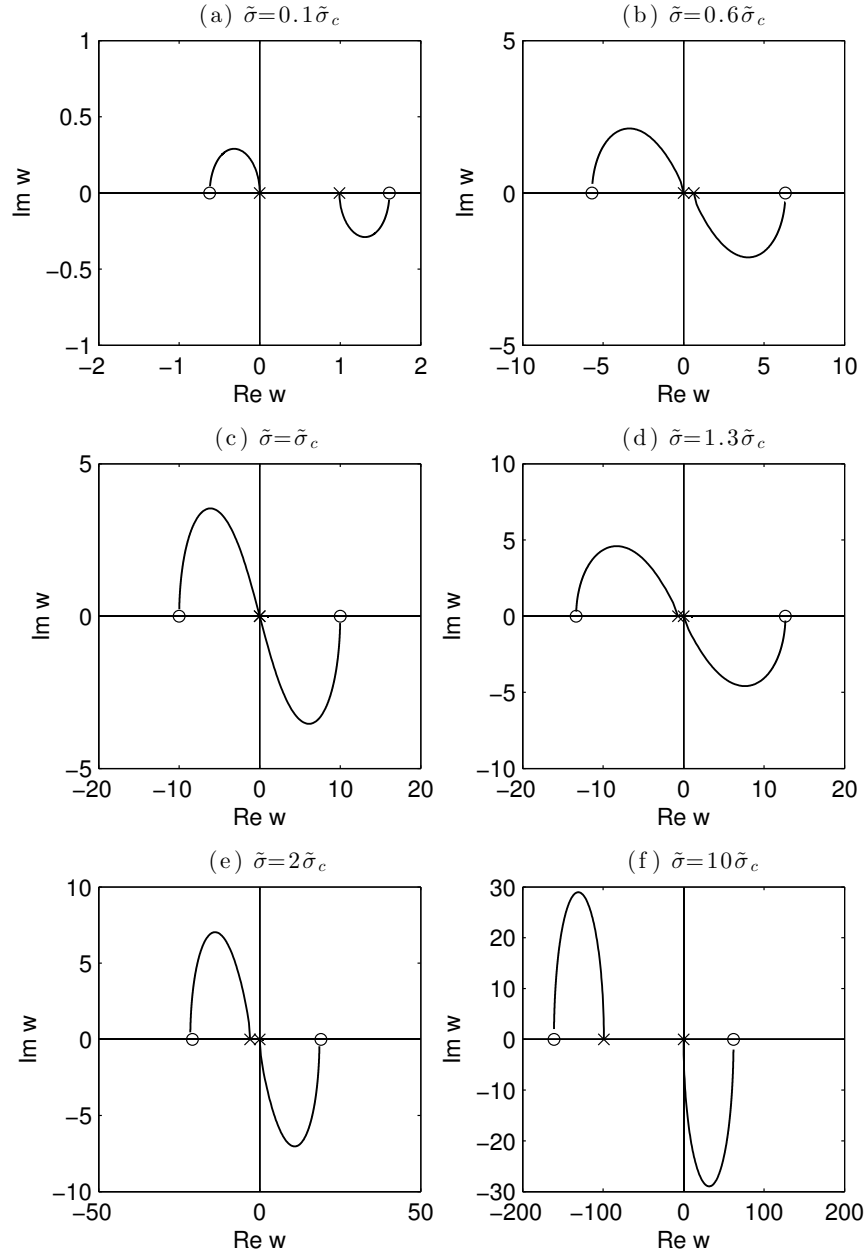


Figure 2.13: The longitudinal squared wave speeds of isotropic thermelastic material for temperature-rate-dependent thermoelasticity theory with incompressibility at uniform temperature (alternative form). For each part ($\varepsilon = 0, \alpha_0 = 0.01, \alpha_1 = 0.02$), (a) $\tilde{\sigma} = 0.1\tilde{\sigma}_c$, (b) $\tilde{\sigma} = 0.6\tilde{\sigma}_c$, (c) $\tilde{\sigma} = \tilde{\sigma}_c$, (d) $\tilde{\sigma} = 1.3\tilde{\sigma}_c$, (e) $\tilde{\sigma} = 2\tilde{\sigma}_c$, (f) $\tilde{\sigma} = 10\tilde{\sigma}_c$.

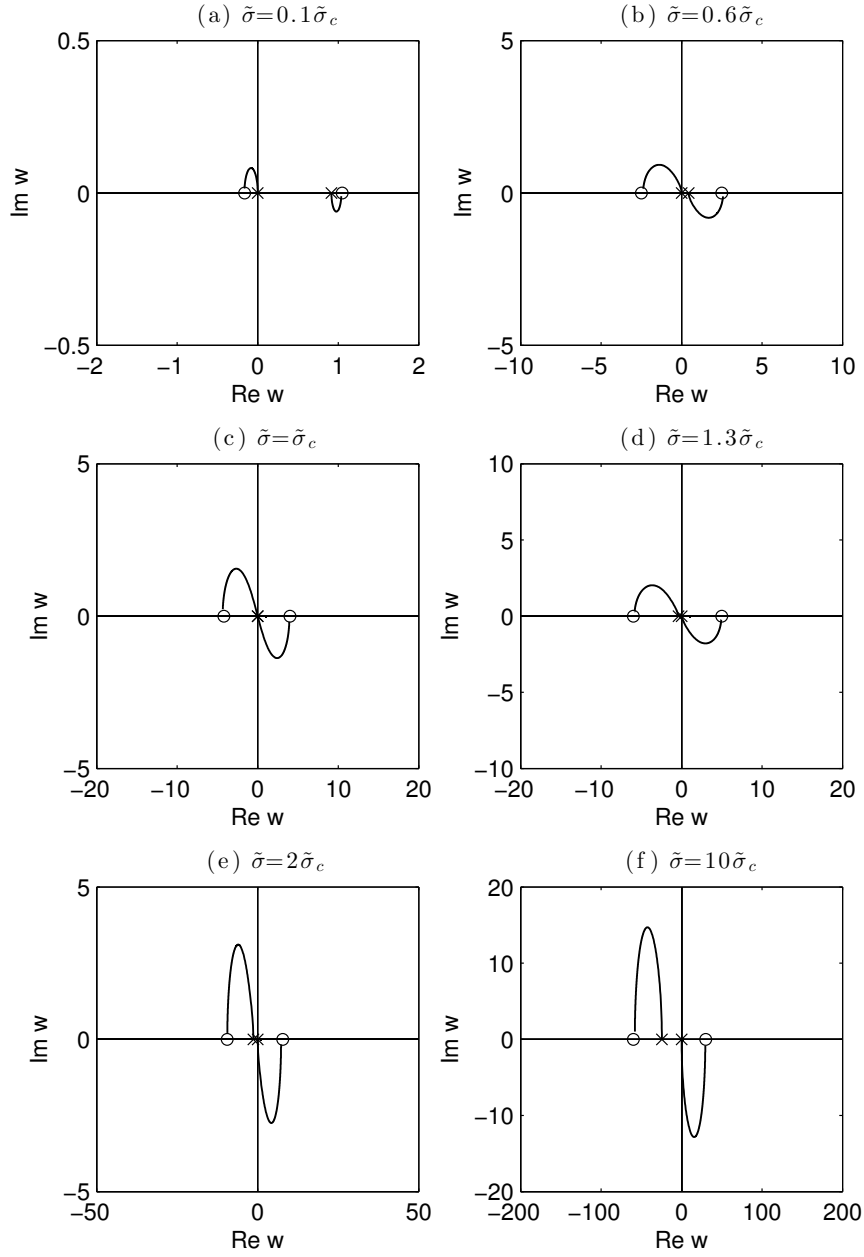


Figure 2.14: The longitudinal squared wave speeds of isotropic thermelastic material for temperature-rate-dependent thermoelasticity theory with incompressibility at uniform temperature. For each part ($\varepsilon = 1, \alpha_0 = 0.01, \alpha_1 = 0.02$), (a) $\tilde{\sigma} = 0.1\tilde{\sigma}_c$, (b) $\tilde{\sigma} = 0.6\tilde{\sigma}_c$, (c) $\tilde{\sigma} = \tilde{\sigma}_c$, (d) $\tilde{\sigma} = 1.3\tilde{\sigma}_c$, (e) $\tilde{\sigma} = 2\tilde{\sigma}_c$, (f) $\tilde{\sigma} = 10\tilde{\sigma}_c$.

Chapter 3

Temperature-rate-dependent thermoelasticity with generalized thermoelasticity: model 1.

Introduction

Chandrasekharaiah and Keshavan [23] combined the field equations of classical thermoelasticity (CTE) and two alternative models of non-classical thermoelasticity, one of the models being Lord and Shulman's theory (GTE), see [24], and the other being Green and Lindsay's theory (TRDTE), see [21]. We shall indicate their theory by the abbreviation TRDTE+GTE (1). In this chapter we consider the anisotropic material and isotropic material separately. Each is either unconstrained or subject to the usual, or an alternative, deformation-temperature constraint. The linearized field equations have been given in each case. We run parallel to Chapter 2 although the analysis here has to include the relaxation time τ . We find that wave stability/instability is affected by the presence of the relaxation times, namely, α_0, α_1 and τ .

3.1 Unconstrained anisotropic TRDTE+GTE (1)

3.1.1 The secular equation

The systems of equations for TRDTE and GTE can be written as follows [23, (2.3)]

$$\left. \begin{aligned} \tilde{c}_{ijkl}u_{k,jl} - \beta_{ij}\left(1 + \alpha_1\frac{\partial}{\partial t}\right)\theta_{,j} &= \rho\ddot{u}_i, \\ k_{ij}\theta_{,ij} - \left(1 + \tau_0\frac{\partial}{\partial t}\right)T\beta_{ij}\dot{u}_{i,j} &= \rho c\left(1 + \alpha_0\frac{\partial}{\partial t}\right)\dot{\theta}. \end{aligned} \right\} \quad (3.1)$$

Putting $\tau_0 = 0$ with $\alpha_1 \geq \alpha_0 \geq 0$ gives TRDTE, and putting $\alpha_1 = 0$ with $\alpha_0 = \tau_0 > 0$, gives GTE. We need to use the form of plane harmonic waves,

$$\{u_i, \theta\} = \{U_i, \Theta\} \exp \{i\omega(s\mathbf{n} \cdot \mathbf{x} - t)\}, \quad (3.2)$$

similarly to Section 2.1 in the previous chapter. Substitute (3.2) into (3.1), by inserting the following derivatives into (3.1),

$$\left. \begin{aligned} u_{k,jl} &= -(\omega s)^2 n_j n_l U_k e^\chi, \\ (\theta + \alpha_1 \dot{\theta})_{,j} &= i\omega s n_j (1 - i\omega \alpha_1) \Theta e^\chi, \quad \ddot{u}_i = U_i (-\omega^2) e^\chi, \\ \theta_{,ij} &= -(\omega s)^2 n_i n_j \Theta e^\chi, \quad \dot{u}_{i,j} = \omega^2 s n_j U_i e^\chi, \\ \ddot{u}_{i,j} &= -i\omega^3 s n_j U_i e^\chi, \quad \dot{\theta} = -i\omega \Theta e^\chi, \quad \ddot{\theta} = -\omega^2 \Theta e^\chi, \end{aligned} \right\} \quad (3.3)$$

where

$$\chi = i\omega(s\mathbf{n} \cdot \mathbf{x} - t).$$

Then cancelling all exponential factors, we obtain the propagation conditions. Firstly (3.1)₁ becomes

$$\tilde{c}_{ijkl}(-\omega^2 s^2) n_j n_l U_k - \beta_{ij}(i(1 - i\omega \alpha_1)\omega s n_j)\Theta = -\rho \omega^2 U_i. \quad (3.4)$$

Rearranging the equation after dividing by $(-\omega^2 s^2)$ we get

$$(\tilde{c}_{ijkl} n_j n_l - \rho s^{-2} \delta_{ik}) U_k + \beta_{ij} n_j (\omega s)^{-1} i(1 - i\omega \alpha_1) \Theta = 0. \quad (3.5)$$

This equation is the same as (2.5)₁ of TRDTE. Now equation (3.1)₂, becomes

$$k_{ij}(-\omega^2 s^2) n_i n_j \Theta - T\beta_{ij}(\omega^2 s n_j U_i + \tau_0(-i\omega^3) s n_j U_i) = \rho c(-i\omega \Theta + \alpha_0(-\omega^2) \Theta). \quad (3.6)$$

Rearranging the equation after dividing by $(-\omega^2 s^2)$, we obtain

$$Ts^{-1}\beta_{ij}n_j(1-i\omega\tau_0)U_i + (k_{ij}n_i n_j - i\omega^{-1}(1-i\omega\alpha_0)c\rho s^{-2})\Theta = 0. \quad (3.7)$$

We have defined earlier that the isothermal acoustic tensor and thermal conductivity scalar are given by

$$\tilde{Q}_{ij} = \tilde{c}_{ijkl}n_i n_j, \quad k = k_{ij}n_i n_j.$$

So, equations (3.5) and (3.7) can be written as

$$\left. \begin{aligned} (\tilde{Q}_{ij} - \rho s^{-2}\delta_{ik})U_k + \beta_{ij}n_j(\omega s)^{-1}i(1-i\alpha_1\omega)\Theta &= 0, \\ Ts^{-1}\beta_{ij}n_j(1-i\omega\tau_0)U_i + (k - i\omega^{-1}(1-i\alpha_0\omega)c\rho s^{-2})\Theta &= 0. \end{aligned} \right\} \quad (3.8)$$

In matrix form, similarly to equations (2.7), on taking

$$\beta_{ij}n_j = b_i, \quad \text{and} \quad \beta = Ts^{-1}(1-i\omega\tau_0),$$

equations (3.8) may be written as

$$\begin{bmatrix} \tilde{Q}_{11} - \rho s^{-2} & \tilde{Q}_{12} & \tilde{Q}_{13} & i(1-i\omega\alpha_1)\omega^{-1}s^{-1}b_1 \\ \tilde{Q}_{21} & \tilde{Q}_{22} - \rho s^{-2} & \tilde{Q}_{23} & i(1-i\omega\alpha_1)\omega^{-1}s^{-1}b_2 \\ \tilde{Q}_{31} & \tilde{Q}_{32} & \tilde{Q}_{33} - \rho s^{-2} & i(1-i\omega\alpha_1)\omega^{-1}s^{-1}b_3 \\ \beta b_1 & \beta b_2 & \beta b_3 & k - i\omega^{-1}(1-i\omega\alpha_0)c\rho s^{-2} \end{bmatrix} \cdot \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ \Theta \end{bmatrix} = 0.$$

These equations have non-zero solutions if and only if

$$\begin{vmatrix} \tilde{Q}_{11} - \rho s^{-2} & \tilde{Q}_{12} & \tilde{Q}_{13} & i(1-i\omega\alpha_1)\omega^{-1}s^{-1}b_1 \\ \tilde{Q}_{21} & \tilde{Q}_{22} - \rho s^{-2} & \tilde{Q}_{23} & i(1-i\omega\alpha_1)\omega^{-1}s^{-1}b_2 \\ \tilde{Q}_{31} & \tilde{Q}_{32} & \tilde{Q}_{33} - \rho s^{-2} & i(1-i\omega\alpha_1)\omega^{-1}s^{-1}b_3 \\ \beta b_1 & \beta b_2 & \beta b_3 & k - i\omega^{-1}(1-i\omega\alpha_0)c\rho s^{-2} \end{vmatrix} = 0.$$

This determinant may be written as

$$D \equiv \begin{vmatrix} \tilde{\mathbf{Q}} - w\mathbf{1} & \bar{\alpha}\mathbf{b} \\ \beta\mathbf{b}^T & \gamma_1 \end{vmatrix}, \quad (3.9)$$

where

$$w = \rho s^{-2}, \quad \bar{\alpha} = i(1-i\alpha_1\omega)\omega^{-1}s^{-1}, \quad \gamma_1 = k - i\omega^{-1}(1-i\omega\alpha_0)cw, \quad \beta = Ts^{-1}(1-i\omega\tau_0). \quad (3.10)$$

We can rewrite this determinant as

$$D \equiv \begin{vmatrix} \tilde{\mathbf{Q}} - w\mathbf{1} & \bar{\alpha}\mathbf{b} + \mathbf{0} \\ \beta\mathbf{b}^T & -\delta + (\gamma_1 + \delta) \end{vmatrix} = 0,$$

in which so far δ is an arbitrary quantity. Using properties of determinants to expand by the fourth column we have

$$D \equiv \begin{vmatrix} \tilde{\mathbf{Q}} - w\mathbf{1} & \bar{\alpha}\mathbf{b} \\ \beta\mathbf{b}^T & -\delta \end{vmatrix} + \begin{vmatrix} \tilde{\mathbf{Q}} - w\mathbf{1} & \mathbf{0} \\ \beta\mathbf{b}^T & \gamma_1 + \delta \end{vmatrix}. \quad (3.11)$$

The first determinant is

$$D_1 = \begin{vmatrix} \tilde{Q}_{11} - w & \tilde{Q}_{12} & \tilde{Q}_{13} & \bar{\alpha}b_1 \\ \tilde{Q}_{21} & \tilde{Q}_{22} - w & \tilde{Q}_{23} & \bar{\alpha}b_2 \\ \tilde{Q}_{31} & \tilde{Q}_{32} & \tilde{Q}_{33} - w & \bar{\alpha}b_3 \\ \beta b_1 & \beta b_2 & \beta b_3 & -\delta \end{vmatrix}.$$

Remove $\bar{\alpha}\mathbf{b}$ from the fourth column by taking

$$\begin{aligned} &\text{row 1} - \left(\frac{\bar{\alpha}b_1}{-\delta}\right) \text{row 4}, \\ &\text{row 2} - \left(\frac{\bar{\alpha}b_2}{-\delta}\right) \text{row 4}, \\ &\text{row 3} - \left(\frac{\bar{\alpha}b_3}{-\delta}\right) \text{row 4}. \end{aligned}$$

So, we obtain

$$D_1 = \begin{vmatrix} (\tilde{Q}_{11} - w) - (\frac{\bar{\alpha}b_1}{-\delta})\beta b_1 & \tilde{Q}_{12} - (\frac{\bar{\alpha}b_1}{-\delta})\beta b_2 & \tilde{Q}_{13} - (\frac{\bar{\alpha}b_1}{-\delta})\beta b_3 & 0 \\ \tilde{Q}_{21} - (\frac{\bar{\alpha}b_2}{-\delta})\beta b_1 & (\tilde{Q}_{22} - w) - (\frac{\bar{\alpha}b_2}{-\delta})\beta b_2 & \tilde{Q}_{23} - (\frac{\bar{\alpha}b_2}{-\delta})\beta b_3 & 0 \\ \tilde{Q}_{31} - (\frac{\bar{\alpha}b_3}{-\delta})\beta b_1 & \tilde{Q}_{32} - (\frac{\bar{\alpha}b_3}{-\delta})\beta b_2 & (\tilde{Q}_{33} - w) - (\frac{\bar{\alpha}b_3}{-\delta})\beta b_3 & 0 \\ \beta b_1 & \beta b_2 & \beta b_3 & -\delta \end{vmatrix}.$$

Expanding D_1 by the fourth column gives

$$D_1 = -\delta \det\{(\tilde{\mathbf{Q}} - w\mathbf{1}) + \frac{\bar{\alpha}\beta}{\delta}\mathbf{b} \otimes \mathbf{b}\}, \quad (3.12)$$

now we may use the relationship between $\tilde{\mathbf{Q}}$ and $\hat{\mathbf{Q}}$, (2.13) and (2.14), to define D_1 in terms of $\hat{\mathbf{Q}}$, with the aid of (3.10)_{2,4} we get the value of δ

$$\delta = i(1 - i\omega\alpha_1)(1 - i\omega\tau_0)\omega^{-1}cw. \quad (3.13)$$

Inserting (3.13) into (3.12), shows that the first determinant is given by

$$D_1 = -i(1 - i\omega\alpha_1)(1 - i\omega\tau_0)\omega^{-1}cw \det\{\hat{\mathbf{Q}} - w\mathbf{1}\}. \quad (3.14)$$

The second determinant of (3.11) is

$$D_2 = \begin{vmatrix} (\tilde{Q}_{11} - w) & \tilde{Q}_{12} & \tilde{Q}_{13} & 0 \\ \tilde{Q}_{21} & (\tilde{Q}_{22} - w) & \tilde{Q}_{23} & 0 \\ \tilde{Q}_{31} & \tilde{Q}_{32} & (\tilde{Q}_{33} - w) & 0 \\ \beta b_1 & \beta b_2 & \beta b_3 & \gamma_1 + \delta \end{vmatrix} = (\gamma_1 + \delta) \det\{\tilde{\mathbf{Q}} - w\mathbf{1}\}. \quad (3.15)$$

So, after inserting (3.10)₃ and (3.13) into (3.15) the second determinant may be written as

$$D_2 = [(k - i\omega^{-1}(1 - i\omega\alpha_0)cw) + i(1 - i\omega\alpha_1)(1 - i\omega\tau_0)\omega^{-1}cw] \det\{\tilde{\mathbf{Q}} - w\mathbf{1}\}. \quad (3.16)$$

Simplifying this equation we get

$$D_2 = [k + cw((\alpha_1 - \alpha_0) + \tau_0(1 - i\omega\alpha_1))] \det\{\tilde{\mathbf{Q}} - w\mathbf{1}\}. \quad (3.17)$$

Thus

$$D \equiv -i(1 - i\omega\alpha_1)(1 - i\omega\tau_0)\omega^{-1}cw \det\{\hat{\mathbf{Q}} - w\mathbf{1}\} + [k + cw((\alpha_1 - \alpha_0) + \tau_0(1 - i\omega\alpha_1))] \det\{\tilde{\mathbf{Q}} - w\mathbf{1}\}. \quad (3.18)$$

Dividing D by $[-i(1 - i\omega\alpha_1)(1 - i\omega\tau_0)\omega^{-1}c]$ we get the secular equation

$$w \det\{\hat{\mathbf{Q}} - w\mathbf{1}\} + \left\{ \frac{i\omega c^{-1}[k + cw((\alpha_1 - \alpha_0) + \tau_0(1 - i\omega\alpha_1))]}{(1 - i\omega\alpha_1)(1 - i\omega\tau_0)} \right\} \det\{\tilde{\mathbf{Q}} - w\mathbf{1}\} = 0. \quad (3.19)$$

This is the secular equation for unconstrained anisotropic TRDTE+GTE (1) and has not previously appeared in the literature.

To non-dimensionlize this equation, we have to use the dimensionless quantities (2.103) with the further non-dimensional quantity

$$\tau = \tau_0\omega^*. \quad (3.20)$$

For a more convenient form we drop the dashes, so the secular equation may be written as

$$w \det\{w\mathbf{1} - \hat{\mathbf{Q}}\} + \left\{ \frac{i\omega[1 + w((\alpha_1 - \alpha_0) + \tau(1 - i\omega\alpha_1))]}{(1 - i\omega\alpha_1)(1 - i\omega\tau)} \right\} \det\{w\mathbf{1} - \tilde{\mathbf{Q}}\} = 0. \quad (3.21)$$

The secular equation can be written in terms of \tilde{q}_i the eigenvalues of $\tilde{\mathbf{Q}}$, and \hat{q}_i , the eigenvalues of $\hat{\mathbf{Q}}$, where $i = 1, 2, 3$ as

$$w(w - \hat{q}_1)(w - \hat{q}_2)(w - \hat{q}_3) + \left\{ \frac{i\omega\{1 + w[(\alpha_1 - \alpha_0) + \tau(1 - i\omega\alpha_1)]\}}{(1 - i\omega\alpha_1)(1 - i\omega\tau)} \right\} (w - \tilde{q}_1)(w - \tilde{q}_2)(w - \tilde{q}_3) = 0. \quad (3.22)$$

Rewrite equation (3.22) in a simple form

$$\hat{F}(w) + \left[\frac{i\omega\{1 + w[(\alpha_1 - \alpha_0) + \tau(1 - i\omega\alpha_1)]\}}{(1 - i\omega\alpha_1)(1 - i\omega\tau)} \right] \tilde{G}(w) = 0. \quad (3.23)$$

We have defined $\hat{F}(w)$ and $\tilde{G}(w)$ earlier in (2.28).

For $\tau = 0$ (3.23) reduces at (2.27), as expected. In the GTE limit, where $\alpha_1 = 0$ and $\alpha_0 = \tau > 0$, the secular equation (3.23) reduces to the anisotropic GTE secular equation, see Scott [32, (4)].

Low frequency expansions

When $\omega = 0$, the roots of the secular equation (3.23) are the zeros of $\hat{F}(w) : w = \hat{q}_i$, $i = 0, 1, 2, 3$, defining, $\hat{q}_0 \equiv 0$. Taylor expansions of the roots of the secular equation (3.23) take the form

$$w_i(\omega) = \hat{q}_i + \sum_{n=1}^{\infty} d_n^{(i)} (-i\omega)^n, \quad i = 0, 1, 2, 3. \quad (3.24)$$

The first branch, when $i = 0$, is

$$w_0(\omega) = \hat{q}_0 + d_1^{(0)} (-i\omega) + O(\omega^2). \quad (3.25)$$

By substituting (3.25) into (3.23), we obtain

$$d_1^{(0)} = \frac{\tilde{q}_1 \tilde{q}_2 \tilde{q}_3}{\hat{q}_1 \hat{q}_2 \hat{q}_3} > 0. \quad (3.26)$$

Insert (3.26) into (3.25), we get

$$w_0(\omega) = -i\omega \frac{\tilde{G}(0)}{\hat{F}'(0)} + O(\omega^2). \quad (3.27)$$

This wave is stable according to the stability criterion (2.24). When $i = 1$, equation (3.24) becomes

$$w_1(\omega) = \hat{q}_1 + d_1^{(1)}(-i\omega) + O(\omega^2). \quad (3.28)$$

Insert (3.28) into (3.23) we get

$$d_1^{(1)} = \{1 + \hat{q}_1(\alpha_1 - \alpha_0 + \tau)\} \frac{(\hat{q}_1 - \tilde{q}_1)(\hat{q}_1 - \tilde{q}_2)(\hat{q}_1 - \tilde{q}_3)}{\hat{q}_1(\hat{q}_1 - \hat{q}_2)(\hat{q}_1 - \hat{q}_3)} > 0, \quad (3.29)$$

because of the interlacing (2.25) and the fact that $\alpha_1 \geq \alpha_0$. Thus,

$$w_1 = \hat{q}_1 - i\omega \{1 + \hat{q}_1(\alpha_1 - \alpha_0 + \tau)\} \frac{\tilde{G}(\hat{q}_1)}{\hat{F}'(\hat{q}_1)} + O(\omega^2). \quad (3.30)$$

It is clear that $\text{Im } w_1(\omega) < 0$, which means the condition of stability (2.24) is satisfied, so this mode is stable. Similarly, when $i = 2, 3$ we find that

$$w_2 = \hat{q}_2 - i\omega \{1 + \hat{q}_2(\alpha_1 - \alpha_0 + \tau)\} \tilde{G}(\hat{q}_2)/\hat{F}'(\hat{q}_2) + O(\omega^2). \quad (3.31)$$

$$w_3 = \hat{q}_3 - i\omega \{1 + \hat{q}_3(\alpha_1 - \alpha_0 + \tau)\} \tilde{G}(\hat{q}_3)/\hat{F}'(\hat{q}_3) + O(\omega^2). \quad (3.32)$$

These two equations also represent stable waves. So there are four stable waves in the low frequency limit.

High frequency expansions

The roots of the secular equation (3.23) in the high frequency limit $\omega \rightarrow \infty$, may be obtained by taking $(i\omega)^{-1} \rightarrow 0$. By putting $\omega = \frac{1}{\zeta}$, so $i\omega = \frac{-1}{i\zeta}$. The secular equation (3.23) becomes

$$\hat{F}(w) - \frac{(i\zeta)^{-1} \{1 + w[(\alpha_1 - \alpha_0) + \tau(1 + (i\zeta)^{-1}\alpha_1)]\}}{(1 + (i\zeta)^{-1}\alpha_1)(1 + (i\zeta)^{-1}\tau)} \tilde{G}(w) = 0. \quad (3.33)$$

We can write (3.33) in the following form

$$\hat{F}(w) + K\tilde{G}(w) = 0, \quad (3.34)$$

where

$$K \equiv \frac{-(i\zeta)^{-1} \{1 + w[(\alpha_1 - \alpha_0) + \tau(1 + (i\zeta)^{-1}\alpha_1)]\}}{(1 + (i\zeta)^{-1}\alpha_1)(1 + (i\zeta)^{-1}\tau)}.$$

Multiplying numerator and denominator of K by $(i\zeta)^2$ we get

$$K = \frac{-\{i\zeta + wi\zeta[\alpha_1 - \alpha_0 + \tau(1 + (i\zeta)^{-1}\alpha_1)]\}}{(i\zeta + \alpha_1)(i\zeta + \tau)}. \quad (3.35)$$

Simplifying,

$$K = \frac{-\{i\zeta + w[i\zeta(\alpha_1 - \alpha_0 + \tau) + \alpha_1\tau]\}}{\alpha_1\tau(1 + i\zeta/\alpha_1)(1 + i\zeta/\tau)}. \quad (3.36)$$

At $\zeta = 0$, (3.36) becomes,

$$K = -w. \quad (3.37)$$

So with aid of (3.37) the secular equation (3.34) evaluated at $\zeta = 0$, is

$$H(w) \equiv w(w - \hat{q}_1)(w - \hat{q}_2)(w - \hat{q}_3) - w(w - \tilde{q}_1)(w - \tilde{q}_2)(w - \tilde{q}_3) = 0. \quad (3.38)$$

This equation is not a quartic in w as the w^4 terms cancel out. It is a cubic equation in w with one root $w = 0$ denoted by $\bar{q}_1 = 0$ and the other roots denoted by \bar{q}_2 and \bar{q}_3 with $\bar{q}_2 < \bar{q}_3$. Now we want to examine the sign changes of $H(w)$. By using (3.38) and the interlacing (2.25) we find that

$$\begin{aligned} H(0) &= 0, \\ H(\tilde{q}_1) &= \tilde{q}_1(\tilde{q}_1 - \hat{q}_1)(\tilde{q}_1 - \hat{q}_2)(\tilde{q}_1 - \hat{q}_3) < 0, \\ H(\hat{q}_1) &= -\hat{q}_1(\hat{q}_1 - \tilde{q}_1)(\hat{q}_1 - \tilde{q}_2)(\hat{q}_1 - \tilde{q}_3) < 0, \\ H(\tilde{q}_2) &= \tilde{q}_2(\tilde{q}_2 - \hat{q}_1)(\tilde{q}_2 - \hat{q}_2)(\tilde{q}_2 - \hat{q}_3) > 0, \\ H(\hat{q}_2) &= -\hat{q}_2(\hat{q}_2 - \tilde{q}_1)(\hat{q}_2 - \tilde{q}_2)(\hat{q}_2 - \tilde{q}_3) > 0, \\ H(\tilde{q}_3) &= \tilde{q}_3(\tilde{q}_3 - \hat{q}_1)(\tilde{q}_3 - \hat{q}_2)(\tilde{q}_3 - \hat{q}_3) < 0, \\ H(\hat{q}_3) &= -\hat{q}_3(\hat{q}_3 - \tilde{q}_1)(\hat{q}_3 - \tilde{q}_2)(\hat{q}_3 - \tilde{q}_3) < 0, \\ H(\infty) &= -\infty < 0, \end{aligned} \quad (3.39)$$

the last following because the coefficient of w^3 in $H(w)$ is

$$-(\hat{q}_1 + \hat{q}_2 + \hat{q}_3 - \tilde{q}_1 - \tilde{q}_2 - \tilde{q}_3) < 0.$$

From (3.39), it is easy to see that the zeros of $H(w)$, are such that the root \bar{q}_2 is between \hat{q}_1 and \tilde{q}_2 and \bar{q}_3 is between \hat{q}_2 and \tilde{q}_3 . It is clear that the zeros $\bar{q}_2 < \bar{q}_3$ of $H(w)$ are real and interlace according to

$$0 < \tilde{q}_1 \leq \hat{q}_1 \leq \bar{q}_2 \leq \tilde{q}_2 \leq \hat{q}_2 \leq \bar{q}_3 \leq \tilde{q}_3 \leq \hat{q}_3. \quad (3.40)$$

Define a cubic polynomial

$$\bar{h}(w) = w(w - \bar{q}_2)(w - \bar{q}_3), \quad (3.41)$$

which must be a scalar multiple of $H(w)$ because both have same three roots:

$$H(w) \equiv -(\hat{q}_1 + \hat{q}_2 + \hat{q}_3 - \tilde{q}_1 - \tilde{q}_2 - \tilde{q}_3)\bar{h}(w). \quad (3.42)$$

Now looking for roots when $\zeta \rightarrow 0$, so we can rewrite (3.35) as follows

$$K = -\left\{w + \frac{i\zeta}{\alpha_1\tau} \left[1 + w(\alpha_1 - \alpha_0 + \tau)\right]\right\} \left(1 - \frac{i\zeta}{\alpha_1}\right) \left(1 - \frac{i\zeta}{\tau}\right) + O(\zeta^2). \quad (3.43)$$

After expanding and simplifying we obtain

$$K = -\left\{w + i\zeta \left(\frac{1}{\alpha_1\tau} - \frac{w\alpha_0}{\alpha_1\tau}\right)\right\} + O(\zeta^2). \quad (3.44)$$

The secular equation (3.34) might written as

$$\hat{F}(w) - \left(w + \frac{i\zeta}{\alpha_1\tau}(1 - \alpha_0 w)\right)\tilde{G}(w) = 0. \quad (3.45)$$

By using (3.38) and (3.42) with (3.45) we get

$$\bar{h}(w) + \frac{i\zeta}{d\alpha_1\tau}(1 - \alpha_0 w)\tilde{G}(w) = 0, \quad (3.46)$$

in which

$$d = \hat{q}_1 + \hat{q}_2 + \hat{q}_3 - \tilde{q}_1 - \tilde{q}_2 - \tilde{q}_3.$$

Based on (2.25) we find that $d > 0$. It is clear that equation (3.46) is a quartic equation in w provided that $\zeta > 0$, so there are four roots w_i , where $i = 1, 2, 3, 4$. Power series expansions of the roots of the secular equation in the high frequency limit take the form

$$w_i(\zeta) = \bar{q}_i + \sum_{n=1}^{\infty} d_n^{(i)}(i\zeta)^n, \quad i = 1, 2, 3, 4. \quad (3.47)$$

The first coefficient, when $i = 1$, is

$$w_1(\zeta) = \bar{q}_1 + d_1^{(1)}(i\zeta) + O(\zeta^2). \quad (3.48)$$

Substituting (3.48) into (3.46), with $\bar{q}_1 \equiv 0$ and $\zeta \rightarrow 0$, we get

$$d_1^{(1)} = \left(\frac{1}{d\alpha_1\tau}\right) \frac{\tilde{q}_1\tilde{q}_2\tilde{q}_3}{\bar{q}_2\bar{q}_3} > 0. \quad (3.49)$$

Inserting (3.49) into (3.48) we get

$$w_1(\zeta) = \left(\frac{i\zeta}{d\alpha_1\tau} \right) \frac{\tilde{G}(0)}{\bar{h}'(0)} + O(\zeta^2). \quad (3.50)$$

In order to write (3.50) in terms of ω , substituting ζ by $(\omega)^{-1}$, we obtain

$$w_1(\omega) = \left(\frac{i\omega^{-1}}{d\alpha_1\tau} \right) \frac{\tilde{G}(0)}{\bar{h}'(0)} + O(\omega^{-2}). \quad (3.51)$$

Clearly, $\text{Im } w_1(\omega) > 0$, thus w_1 is unstable.

When $i = 2$, (3.47) becomes

$$w_2(\zeta) = \bar{q}_2 + d_1^{(2)}(i\zeta) + O(\zeta^2). \quad (3.52)$$

Inserting (3.52) into (3.46) we get

$$d_1^{(2)} = \frac{-(1 - \alpha_0\bar{q}_2)(\bar{q}_2 - \tilde{q}_1)(\bar{q}_2 - \tilde{q}_2)(\bar{q}_2 - \tilde{q}_3)}{d\alpha_1\tau\bar{q}_2(\bar{q}_2 - \bar{q}_3)}. \quad (3.53)$$

Substituting (3.53) into (3.52) we get

$$w_2(\zeta) = \bar{q}_2 - \left(\frac{i\zeta(1 - \alpha_0\bar{q}_2)}{d\alpha_1\tau} \right) \frac{\tilde{G}(\bar{q}_2)}{\bar{h}'(\bar{q}_2)} + O(\zeta^2). \quad (3.54)$$

The sign of $d_1^{(2)}$ depends on the sign of $(1 - \alpha_0\bar{q}_2)$. If $(1 - \alpha_0\bar{q}_2)$ is negative, which means

$$1 - \alpha_0\bar{q}_2 < 0, \quad \Rightarrow \quad 1 < \alpha_0\bar{q}_2, \quad \Rightarrow \quad \bar{q}_2 > 1/\alpha_0,$$

or

$$\alpha_0 > 1/\bar{q}_2,$$

then $d_1^{(2)} < 0$, so that $w_2(\zeta)$ is stable.

Rewrite (3.54) in terms of ω

$$w_2(\omega) = \bar{q}_2 - \left(\frac{i\omega^{-1}(1 - \alpha_0\bar{q}_2)}{d\alpha_1\tau} \right) \frac{\tilde{G}(\bar{q}_2)}{\bar{h}'(\bar{q}_2)} + O(\omega^{-2}). \quad (3.55)$$

So the stability of $w_2(\omega)$ is satisfied when $\alpha_0 > 1/\bar{q}_2$ or $\bar{q}_2 > 1/\alpha_0$.

Similarly when $i = 3$, we find that

$$d_1^{(3)} = \frac{-(1 - \alpha_0\bar{q}_3)(\bar{q}_3 - \tilde{q}_1)(\bar{q}_3 - \tilde{q}_2)(\bar{q}_3 - \tilde{q}_3)}{d\alpha_1\tau\bar{q}_3(\bar{q}_3 - \bar{q}_2)}, \quad (3.56)$$

So,

$$w_3(\omega) = \bar{q}_3 - \left(\frac{i\omega^{-1}(1 - \alpha_0\bar{q}_3)}{d\alpha_1\tau} \right) \frac{\tilde{G}(\bar{q}_3)}{\bar{h}'(\bar{q}_3)} + O(\omega^{-2}). \quad (3.57)$$

Again, the stability of $w_3(\omega)$ is satisfied when $\alpha_0 > 1/\bar{q}_3$ or $\bar{q}_3 > 1/\alpha_0$.

The fourth root, which is large when ζ is small, can be written as

$$w_4 = B(i\zeta)^{-1} + A + O(\zeta), \quad (3.58)$$

where A and B are constants. By substituting (3.58) into (3.46) we get

$$\begin{aligned} \{A + B(i\zeta)^{-1}\} \{A - \bar{q}_2 + B(i\zeta)^{-1}\} \{A - \bar{q}_3 + B(i\zeta)^{-1}\} + \frac{i\zeta}{d\alpha_1\tau} \{1 - \alpha_0 A - \alpha_0 B(i\zeta)^{-1}\} \\ \{A - \tilde{q}_1 + B(i\zeta)^{-1}\} \{A - \tilde{q}_2 + B(i\zeta)^{-1}\} \{A - \tilde{q}_3 + B(i\zeta)^{-1}\} = 0. \end{aligned} \quad (3.59)$$

Multiply by $(i\zeta)^3$ to obtain

$$\begin{aligned} \{A(i\zeta) + B\} \{(A - \bar{q}_2)(i\zeta) + B\} \{(A - \bar{q}_3)(i\zeta) + B\} + \frac{1}{d\alpha_1\tau} \{(1 - \alpha_0 A)(i\zeta) - \alpha_0 B\} \\ \{(A - \tilde{q}_1)(i\zeta) + B\} \{(A - \tilde{q}_2)(i\zeta) + B\} \{(A - \tilde{q}_3)(i\zeta) + B\} = 0. \end{aligned} \quad (3.60)$$

Putting $\zeta = 0$, we get

$$\begin{aligned} B^3 + \frac{1}{d\alpha_1\tau} (-\alpha_0) B^4 = 0, \\ \Rightarrow B^3 \left(1 - \frac{\alpha_0}{d\alpha_1\tau} B \right) = 0. \end{aligned}$$

Then we obtain

$$B = 0, 0, 0, \frac{d\alpha_1\tau}{\alpha_0}. \quad (3.61)$$

Expanding and simplifying (3.60), ignoring high powers of $(i\zeta)$,

$$\begin{aligned} A(i\zeta)B^2 + (A - \bar{q}_2)(i\zeta)B^2 + (A - \bar{q}_3)(i\zeta)B^2 + B^3 + \frac{1}{d\alpha_1\tau} \left[(1 - \alpha_0 A)(i\zeta)B^3 \right. \\ \left. + (A - \tilde{q}_1)(i\zeta)(-\alpha_0 B)B^2 + (A - \tilde{q}_2)(i\zeta)(-\alpha_0 B)B^2 + (A - \tilde{q}_3)(i\zeta)(-\alpha_0 B)B^2 - \alpha_0 B^4 \right] = 0. \end{aligned} \quad (3.62)$$

Coefficients of $i\zeta$ are

$$AB^2 + (A - \bar{q}_2)B^2 + (A - \bar{q}_3)B^2 + \frac{1}{d\alpha_1\tau} [1 - \alpha_0 A - \alpha_0(A - \tilde{q}_1) - \alpha_0(A - \tilde{q}_2) - \alpha_0(A - \tilde{q}_3)] B^3 = 0.$$

Cancel B^2 as $B \neq 0$ to get

$$3A - \bar{q}_2 - \bar{q}_3 + \frac{1}{d\alpha_1\tau}[1 - \alpha_0(4A - \tilde{q}_1 - \tilde{q}_2 - \tilde{q}_3)]B = 0.$$

By using (3.61)₄, we obtain

$$A = \frac{1}{\alpha_0} + \tilde{q}_1 + \tilde{q}_2 + \tilde{q}_3 - \bar{q}_2 - \bar{q}_3. \quad (3.63)$$

Substituting A and B into (3.58) and write it in terms of ω we obtain

$$w_4(\omega) = \left[\frac{1}{\alpha_0} + \tilde{q}_1 + \tilde{q}_2 + \tilde{q}_3 - \bar{q}_2 - \bar{q}_3 \right] - i\omega \left(\frac{d\alpha_1\tau}{\alpha_0} \right) + O(\omega^{-1}). \quad (3.64)$$

$\text{Im } w_4(\omega) < 0$, thus w_4 is stable.

We note that now branches can change their stability nature for intermediate frequencies, as is illustrated in Figures 3.1–3.4. This is because (3.23) cannot be rearranged to have w on one side and $i\omega$ on the other, as in (2.46).

Numerical results

In each of Figures 3.1–3.4 we used the same values of \hat{q}_i and \tilde{q}_i , $i = 1, 2, 3$, and various choices for τ , except that in Figure 3.4 we made τ a constant. In Figures 3.1 and 3.2, where $\alpha_1 > \alpha_0 > 0$ and $\alpha_1 > \alpha_0 = 0$, respectively, we have three unstable waves and one stable. In Figure 3.3, $\alpha_1 = \alpha_0 > 0$, which gives three stable waves and one unstable. In Figure 3.4 we fixed τ and α_1 and made different choices for α_0 and we found that: in sub-figures (a) and (b), with $\alpha_0 = 0.1$ and 0.3 , respectively, there are three unstable waves and one stable; in sub-figure (c), with $\alpha_0 = 0.5$, there are two stable waves and two unstable; in sub-figures (d)–(f), with $\alpha_0 = 1, 1.5, 2$, sequentially, there are three stable waves and one unstable. Therefore, increasing α_0 affects wave stability.

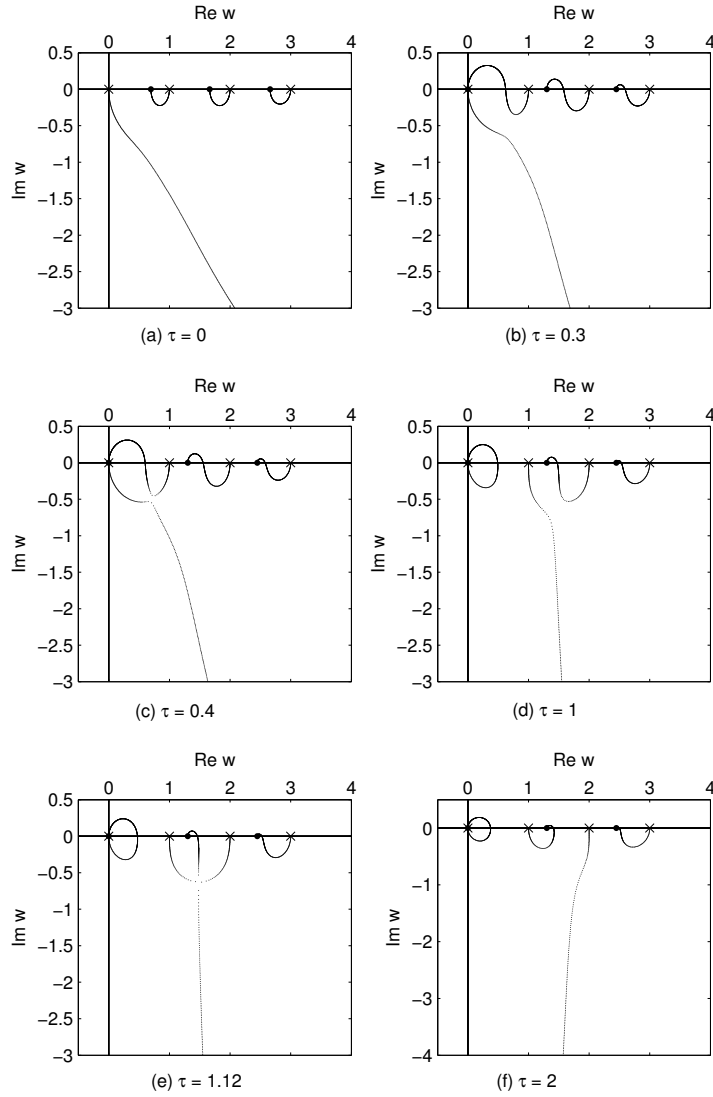


Figure 3.1: The four branches of the secular equation for unconstrained anisotropic thermelastc material for Chandrasakharaiiah and Keshavan's theory. For each part, $\alpha_0 = 0.1$, $\alpha_1 = 0.2$, $\tilde{q}_1 = 0.75$, $\tilde{q}_2 = 1.75$, $\tilde{q}_3 = 2.75$, $\hat{q}_1 = 1$, $\hat{q}_2 = 2$, $\hat{q}_3 = 3$.

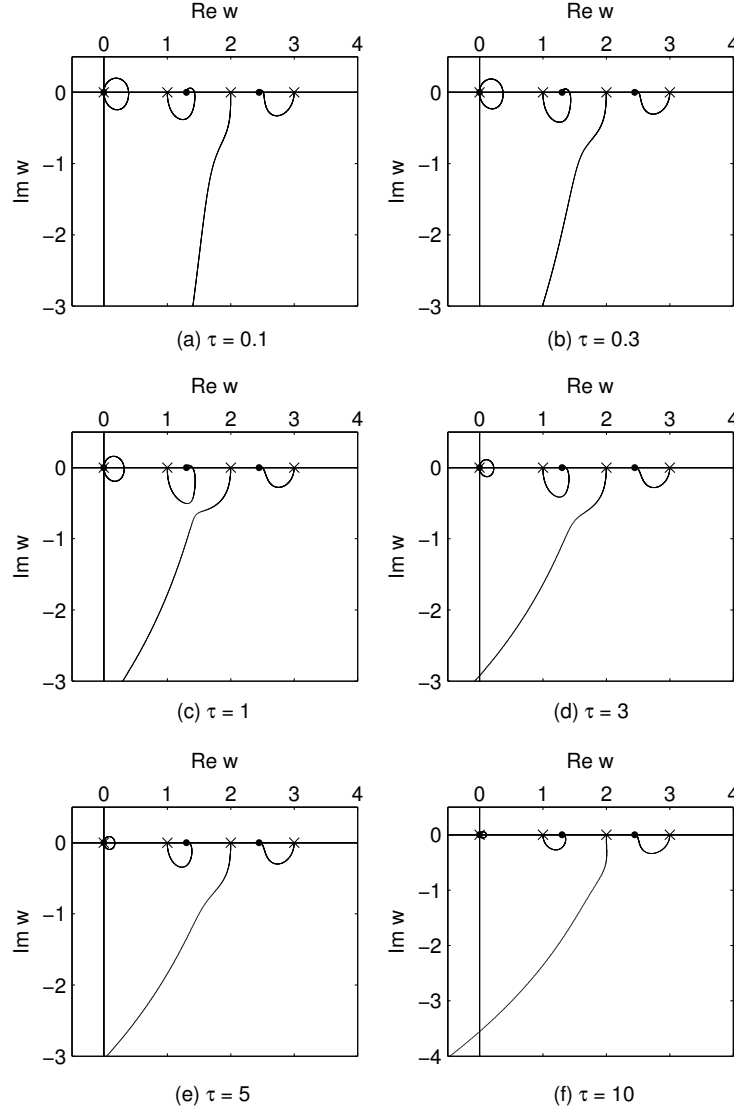


Figure 3.2: The four branches of the secular equation for unconstrained anisotropic thermelastical material for Chandrasekharaiah and Keshavan's theory. For each part, $\alpha_0 = 0$, $\alpha_1 = 2$, $\tilde{q}_1 = 0.75$, $\tilde{q}_2 = 1.75$, $\tilde{q}_3 = 2.75$, $\hat{q}_1 = 1$, $\hat{q}_2 = 2$, $\hat{q}_3 = 3$.

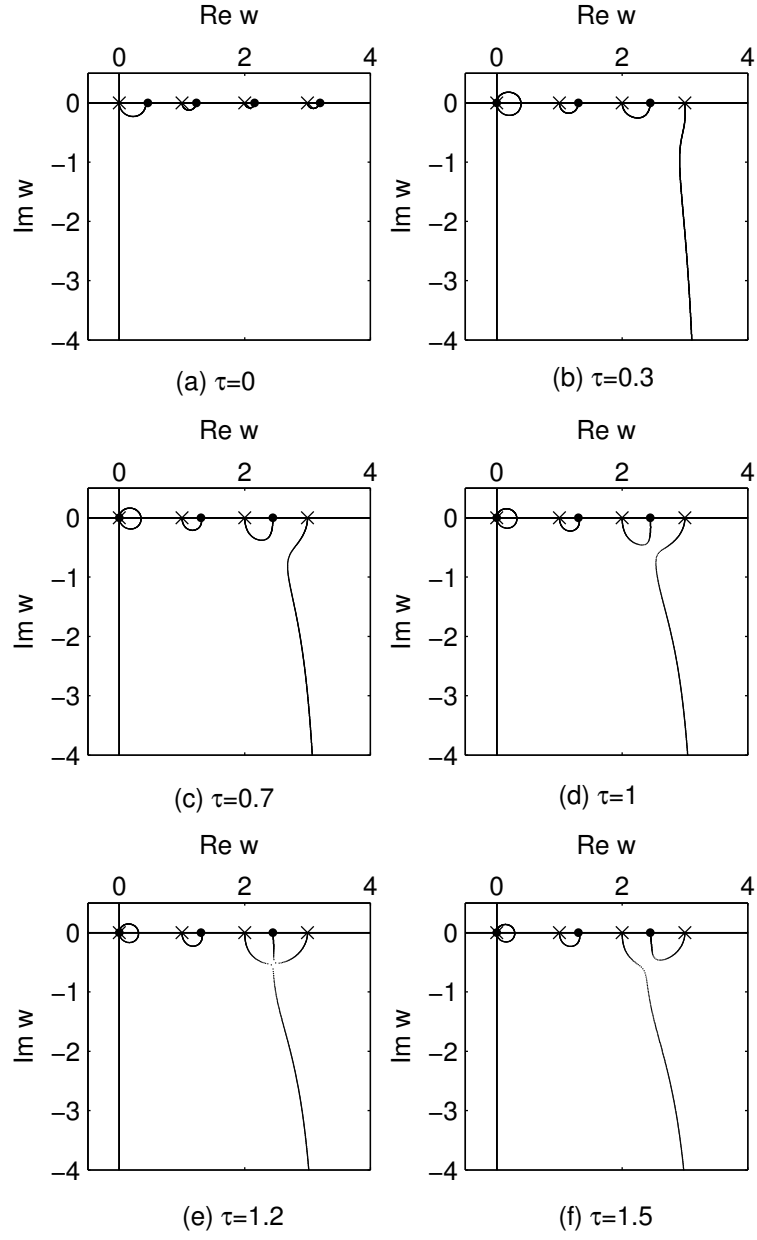


Figure 3.3: The four branches of the secular equation for unconstrained anisotropic thermelastic material for Chandrasekharaiah and Keshavan's theory. For each part, $\alpha_0 = \alpha_1 = 1$, $\tilde{q}_1 = 0.75$, $\tilde{q}_2 = 1.75$, $\tilde{q}_3 = 2.75$, $\hat{q}_1 = 1$, $\hat{q}_2 = 2$, $\hat{q}_3 = 3$.

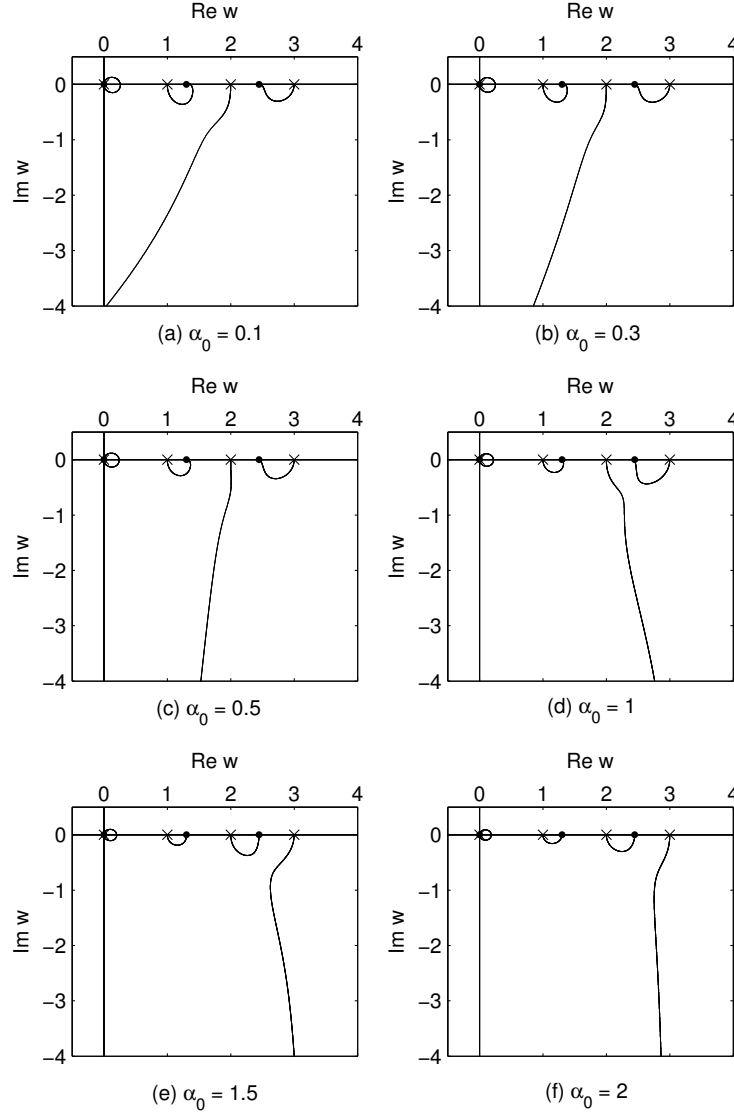


Figure 3.4: The four branches of the secular equation for unconstrained anisotropic thermelastical material for Chandrasekharaiah and Keshavan's theory. For each part, $\tau = 1, \alpha_1 = 3, \tilde{q}_1 = 0.75, \tilde{q}_2 = 1.75, \tilde{q}_3 = 2.75, \hat{q}_1 = 1, \hat{q}_2 = 2, \hat{q}_3 = 3$.

3.2 Unconstrained isotropic TRDTE+GTE (1)

Applying equation (2.48) to (3.1) gives the field equations of isotropic unconstrained thermoelasticity:

$$\left. \begin{aligned} (\tilde{\lambda} + \tilde{\mu})u_{j,ij} + \tilde{\mu}u_{i,jj} - \beta(\theta + \alpha_1\dot{\theta})_{,i} &= \rho\ddot{u}_i, \\ k\theta_{,ii} - \left(1 + \tau\frac{\partial}{\partial t}\right)T\beta\dot{u}_{j,j} &= \rho c(\dot{\theta} + \alpha_0\ddot{\theta}). \end{aligned} \right\} \quad (3.65)$$

3.2.1 The secular equation

Similarly to Section 2.2 we seek solutions of (3.65) in the form of plane harmonic waves (3.2). Insert (3.2) into (3.65) and cancel the exponential factors. From (3.65)₁, we find that

$$[(\tilde{\lambda} + \tilde{\mu})n_in_j + (\tilde{\mu} - \rho s^{-2})\delta_{ij}]U_j + i\beta(\omega s)^{-1}n_i(1 - i\omega\alpha_1)\Theta = 0. \quad (3.66)$$

It is similar to (2.52). From (3.65)₂ we find that

$$k(-(\omega s)^2n_in_i\Theta) - T\beta\omega^2sn_j(1 - i\omega\tau)U_j = \rho c(-i\omega)(1 - i\omega\alpha_0)\Theta.$$

Dividing this equation by $(-\omega)$ then rearranging we get

$$T\beta\omega s(1 - i\omega\tau)n_jU_j + (k\omega s^2 - i\rho c(1 - i\omega\alpha_0))\Theta = 0. \quad (3.67)$$

To eliminate Θ between (3.66) and (3.67), we need to rewrite (3.67) as

$$\Theta = \frac{-T\omega s(1 - i\omega\tau)\beta n_jU_j}{\omega s^2k - i\rho c(1 - i\omega\alpha_0)}. \quad (3.68)$$

Inserting (3.68) into (3.66) we obtain

$$\left((\tilde{\lambda} + \tilde{\mu})n_in_j + (\tilde{\mu} - \rho s^{-2})\delta_{ij}\right)U_j - \left(\frac{i\beta^2T(1 - i\omega\alpha_1)(1 - i\omega\tau)n_in_j}{\omega s^2k - i\rho c(1 - i\omega\alpha_0)}\right)U_j = 0.$$

Rearranging the equation we get

$$\left\{(\tilde{\mu} - w)\delta_{ij} + (\tilde{\lambda} + \tilde{\mu})n_in_j + \frac{wT\beta^2(1 - i\omega\alpha_1)(1 - i\omega\tau)n_in_j}{\rho c(w(1 - i\omega\alpha_0) + i\omega k/c)}\right\}U_j = 0, \quad (3.69)$$

where $w = \rho s^{-2}$. Using the dimensionless quantities (2.57) we get the non-dimensional form of (3.69) as follows

$$\det \left\{ (\tilde{\mu}' - w') \mathbf{1} + \left[\tilde{\lambda}' + \tilde{\mu}' + \frac{\varepsilon w' (1 - i\omega' \alpha'_1) (1 - i\omega' \tau')}{w' (1 - i\omega' \alpha'_0) + i\omega'} \right] \mathbf{n} \otimes \mathbf{n} \right\} = 0. \quad (3.70)$$

Again, using the standard identity (2.60), dropping the dashes for convenience, we get

$$(w - \tilde{\mu})^2 \{ w^2 (1 - i\omega \alpha_0) - w [(1 - i\omega \alpha_0) (\tilde{\lambda} + 2\tilde{\mu}) + \varepsilon (1 - i\omega \alpha_1) (1 - i\omega \tau) - i\omega] - i\omega (2\tilde{\mu} + \tilde{\lambda}) \} = 0. \quad (3.71)$$

This is the secular equation for unconstrained isotropic TRDTE+GTE (1) and has not previously appeared in the literature.

The repeated root $w = \tilde{\mu}$ of (3.71) corresponds to two transverse waves. The quadratic factor of (3.71) is

$$w^2 (1 - i\omega \alpha_0) - w [(1 - i\omega \alpha_0) + \varepsilon (1 - i\omega \alpha_1) (1 - i\omega \tau) - i\omega] - i\omega = 0. \quad (3.72)$$

where

$$\tilde{\lambda} + 2\tilde{\mu} = 1.$$

In the GTE limit, where $\alpha_1 = 0$ and $\alpha_0 = \tau > 0$, the secular equation (3.72) reduces to the isotropic GTE secular equation, see Leslie and Scott [33, (2.14)]. For $\tau = 0$ (3.72) reduces to (2.70), as expected.

The roots of this quadratic equation (3.72) are given by

$$w_{1,2} = \frac{1}{2(1 - i\omega \alpha_0)} [z_2 \pm (z_2^2 + 4i\omega(1 - i\omega \alpha_0))^{\frac{1}{2}}]. \quad (3.73)$$

in which,

$$z_2 = (1 - i\omega \alpha_0) + \varepsilon (1 - i\omega \alpha_1) (1 - i\omega \tau) - i\omega. \quad (3.74)$$

This first root is

$$w_1 = \frac{1}{2(1 - i\omega \alpha_0)} [z_2 + (z_2^2 + 4i\omega(1 - i\omega \alpha_0))^{\frac{1}{2}}], \quad (3.75)$$

where

$$z_2^2 = ((1 - i\omega \alpha_0) + \varepsilon (1 - i\omega \alpha_1) (1 - i\omega \tau))^2 - 2i\omega \varepsilon (1 - i\omega \alpha_1) (1 - i\omega \tau) - 2i\omega (1 - i\omega \alpha_0) - \omega^2.$$

The roots (3.73) can be plotted for varying values of ε , the measure of the degree of thermoelastic coupling, as shown in Figure 3.5. In the uncoupled case, when $\varepsilon = 0$, the roots of (3.73) reduce to

$$w_1 = 1, \quad w_2 = \frac{-i\omega}{(1 - i\omega\alpha_0)}, \quad (3.75a)$$

where w_1 represents an unattenuated, non-dispersive longitudinal wave (a purely elastic mode) and w_2 represents a diffusive mode.

Now we investigate the nature of the modes at high and low frequencies for $\varepsilon > 0$.

Low frequency expansions

For $\omega = 0$ the roots of (3.73) are

$$w_1 = 1 + \varepsilon, \quad w_2 = 0.$$

The roots of the secular equation (3.72) as $\omega \rightarrow 0$ may be written as

$$w_1 = (1 + \varepsilon) + A(i\omega) + O(\omega^2), \quad w_2 = B(i\omega) + O(\omega^2). \quad (3.76)$$

By inserting (3.76) into (3.72) we get

$$A = -\varepsilon[\alpha_1 - \alpha_0 + \tau + \frac{1}{1 + \varepsilon}],$$

and

$$B = \frac{-1}{1 + \varepsilon}.$$

Now the roots are

$$w_1 = (1 + \varepsilon) - i\omega\varepsilon[\alpha_1 - \alpha_0 + \tau + \frac{1}{1 + \varepsilon}] + O(\omega^2).$$

and

$$w_2 = \frac{-i\omega}{1 + \varepsilon} + O(\omega^2).$$

It is clear that $\text{Im } w_1 < 0$, and $\text{Im } w_2 < 0$, so w_1 and w_2 are stable in the low frequency limits.

High frequency expansions

In the high frequency $\omega \rightarrow \infty$, i.e. $\frac{1}{\omega} \rightarrow 0$. So to get the roots of the secular equation (3.72) in the high frequency we need first to divide (3.72) by $(i\omega)^2$ as

$$w^2 \left(\frac{1}{(i\omega)^2} - \frac{\alpha_0}{i\omega} \right) - w \left[\frac{1}{(i\omega)^2} - \frac{\alpha_0}{i\omega} + \varepsilon \left(\frac{1}{(i\omega)^2} - \frac{1}{i\omega} (\alpha_1 + \tau) - \alpha_1 \tau \right) - \frac{1}{i\omega} \right] - \frac{1}{i\omega} = 0. \quad (3.77)$$

Putting $\frac{1}{i\omega} = 0$, equation (3.77) becomes

$$w\alpha_1\tau\varepsilon = 0.$$

The roots are

$$w_1 = 0, \quad w_2 \rightarrow \infty. \quad (3.78)$$

Now we are looking for the roots as $\frac{1}{i\omega} \rightarrow 0$, so the roots may be written as

$$w_1 = A(i\omega)^{-1} + O(\omega^{-2}), \quad w_2 = B(i\omega) + C + O(\omega^{-1}). \quad (3.79)$$

After Substituting (3.79) into (3.77) we obtain

$$A = \frac{1}{\varepsilon\alpha_1\tau}, \quad B = \frac{\alpha_1\tau\varepsilon}{\alpha_0}, \quad \text{and} \quad C = \frac{1}{\alpha_0} \left[\frac{\alpha_1\tau\varepsilon}{\alpha_0} + \varepsilon(\alpha_1 + \tau) + (\alpha_0 + 1) \right]. \quad (3.80)$$

Now roots become

$$w_1 = \frac{-i\omega^{-1}}{\varepsilon\alpha_1\tau} + O(\omega^{-2}), \quad w_2 = \frac{i\omega\alpha_1\tau\varepsilon}{\alpha_0} + \frac{1}{\alpha_0} \left[\frac{\alpha_1\tau\varepsilon}{\alpha_0} + \varepsilon(\alpha_1 + \tau) + (\alpha_0 + 1) \right] + O(\omega^{-1}).$$

It can be seen that $\text{Im } w_1(\omega) < 0$, so w_1 is stable in the high frequency and $\text{Im } w_2(\omega) > 0$ thus w_2 is unstable in the high frequency.

Numerical results

We can plot the roots (3.73) for different values of ε , the measure of the degree of thermoelastic coupling. When $\varepsilon = 0$, the uncoupled case, as shown in Figure 3.5(a), equation (3.75a) is satisfied, and one branch is stable and the other degenerates to single point. When $\varepsilon > 0$ there is always one stable branch tending to infinity and an unstable branch ending at the origin as illustrated in the other sub-figures. It is clear from Figure 3.5 that w_1 is elastic and w_2 is diffusive in character when $\varepsilon = 0$ as we see in (a), but in the other parts (b)–(f) w_1 remains elastic and is accordingly marked with both \times in the low frequency and \circ in the high frequency. But w_2 propagates with the dimensionless squared wave speed $1 + \varepsilon$ which marked \times when $\omega \rightarrow 0$ in each part of Figure 3.5, but as $\omega \rightarrow \infty$, w_2 is diffusive. Both branches are stable in the low frequency but in the high frequency one remains stable but the other becomes unstable.

Varying the parameters α_1 , α_0 and τ while changing the magnitude of ω does not have any substantive influence on the stability.

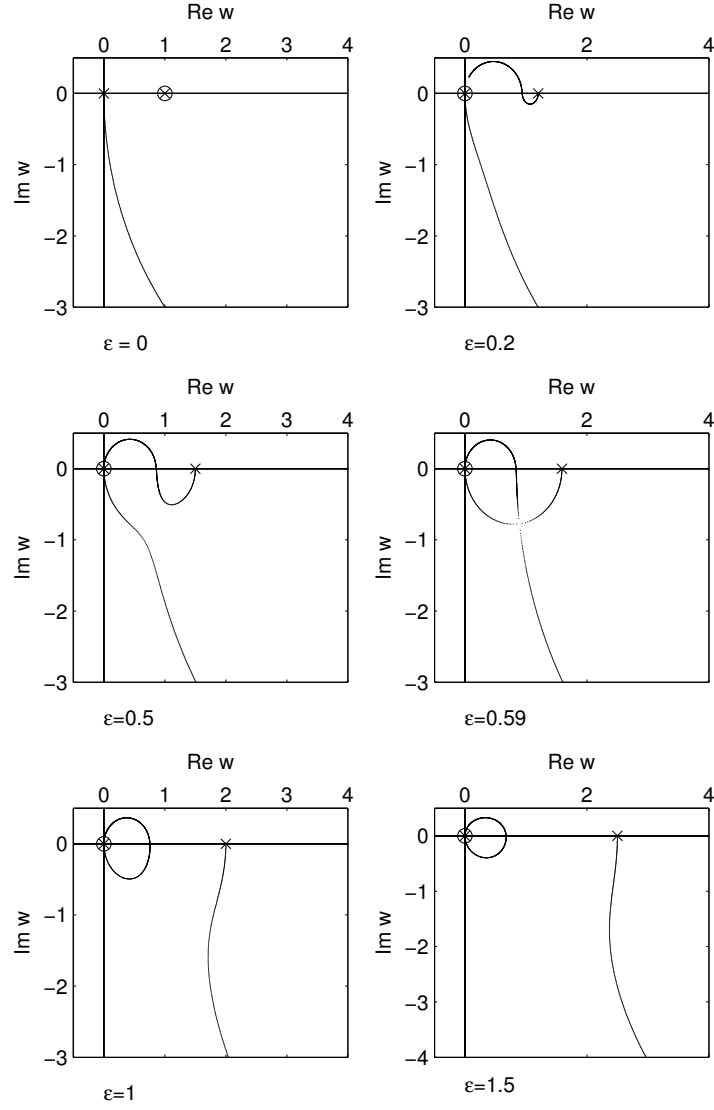


Figure 3.5: The longitudinal squared wave speeds of unconstrained isotropic TRDTE + GTE (1). For each part, $\alpha_0 = 0.1$, $\alpha_1 = 0.2$, $\tau = 0.1$.

3.3 Constrained anisotropic TRDTE+GTE (1)

3.3.1 Usual form of deformation-temperature constraint

The field equations of combined temperature-rate-dependent thermoelasticity and generalized thermoelasticity for a deformation-temperature constrained anisotropic material are

$$\left. \begin{aligned} \tilde{c}_{ijkl}u_{k,lj} - \beta_{ij}\left(1 + \alpha_1\frac{\partial}{\partial t}\right)\theta_{,j} + \tilde{\eta}_{,j}\tilde{N}_{ij} &= \rho\ddot{u}_i, \\ k_{ij}\theta_{,ij} - \left(1 + \tau\frac{\partial}{\partial t}\right)T\beta_{ij}\dot{u}_{i,j} &= \rho c\left(1 + \alpha_0\frac{\partial}{\partial t}\right)\dot{\theta} + T\alpha\dot{\tilde{\eta}}, \\ \tilde{N}_{qp}u_{p,q} - \alpha\theta &= 0. \end{aligned} \right\} \quad (3.81)$$

Equation (3.81)₁ is similar to (2.83), equation (3.81)₂ by analogy to (2.85) and (3.81)₃ is the same constraint (2.81).

The secular equation

Similarly to Section 2.3.1 we find solutions for (3.81) in the form of plane harmonic waves of equation (2.86). We have already found most of the derivatives in (3.81), now we just need to find the rest as in the following

$$\ddot{u}_{i,j} = -i\omega^3 s n_j U_i, \quad \dot{\tilde{\eta}} = -i\omega \tilde{H}.$$

Substituting all derivatives into (3.81)₁ we get

$$(\tilde{Q}_{ik} - \rho s^{-2} \delta_{ik})U_k + i(\omega s)^{-1}[b_i(1 - i\omega\alpha_1)\Theta - \tilde{c}_i\tilde{H}] = 0. \quad (3.82)$$

This equation is similar to (2.89). From (3.81)₂, after inserting derivatives, we find that

$$k_{ij}(-(\omega s)^2 n_i n_j)\Theta - T\beta_{ij}(\omega^2 s n_j U_i + \tau(-i\omega^3 s n_j)U_i) = \rho c(-i\omega\Theta + \alpha_0(-\omega^2)\Theta) + T\alpha(-i\omega\tilde{H}).$$

Rearranging the equation after dividing by $(-\omega)$ we obtain

$$\omega s T b_p (1 - i\omega\tau)U_p - i\alpha T \tilde{H} + (\omega s^2 k - i\rho c(1 - i\omega\alpha_0))\Theta = 0. \quad (3.83)$$

After substituting the derivatives into (3.81)₃ we will have the same equation (2.92)

$$i\omega s \tilde{c}_p U_p - \alpha\Theta = 0. \quad (3.84)$$

In order to eliminate Θ and \tilde{H} between equations (3.82), (3.83) and (3.84), we have to rewrite (3.84) as

$$\Theta = \alpha^{-1}i\omega s\tilde{c}_p U_p. \quad (3.85)$$

On substituting (3.85) into (3.83) it is readily established that

$$\tilde{H} = (\alpha^2 T)^{-1}\omega s(\omega s^2 k - i\rho_e c(1 - i\omega\alpha_0))\tilde{c}_p U_p - i\omega s\alpha^{-1}b_p(1 - i\omega\tau)U_p. \quad (3.86)$$

Inserting (3.85) and (3.86) into (3.82) we obtain

$$\begin{aligned} \{\tilde{Q}_{ip} - \alpha^{-1}(b_i\tilde{c}_p(1 - i\omega\alpha_1) + \tilde{c}_i b_p(1 - i\omega\tau)) - (\alpha^2 T)^{-1}(i\omega s^2 k + \rho c(1 - i\omega\alpha_0))\tilde{c}_i\tilde{c}_p \\ - \rho s^{-2}\delta_{ip}\}U_p = 0. \end{aligned} \quad (3.87)$$

By expanding this equation, we get

$$\begin{aligned} \{\tilde{Q}_{ip} - \alpha^{-1}(b_i\tilde{c}_p + \tilde{c}_i b_p) - (\alpha^2 T)^{-1}(i\omega s^2 k + \rho c(1 - i\omega\alpha_0))\tilde{c}_i\tilde{c}_p + \alpha^{-1}i\omega(\alpha_1 b_i\tilde{c}_p + \tau\tilde{c}_i b_p) \\ - \rho s^{-2}\delta_{ip}\}U_p = 0. \end{aligned}$$

The non-zero amplitudes U_p satisfy (3.82), (3.83) and (3.84) if and only if

$$\begin{aligned} \det\{\tilde{\mathbf{Q}} - \alpha^{-1}(\mathbf{b} \otimes \tilde{\mathbf{c}} + \tilde{\mathbf{c}} \otimes \mathbf{b}) - (\alpha^2 T)^{-1}(\rho c)\tilde{\mathbf{c}} \otimes \tilde{\mathbf{c}} - (\alpha^2 T)^{-1}(i\omega s^2 k - i\omega\alpha_0\rho c)\tilde{\mathbf{c}} \otimes \tilde{\mathbf{c}} \\ + \alpha^{-1}i\omega[\alpha_1\mathbf{b} \otimes \tilde{\mathbf{c}} + \tau\tilde{\mathbf{c}} \otimes \mathbf{b}] - \rho s^{-2}\mathbf{1}\} = 0. \end{aligned} \quad (3.88)$$

By defining

$$\tilde{\mathbf{P}} := \tilde{\mathbf{Q}} - \alpha^{-1}(\mathbf{b} \otimes \tilde{\mathbf{c}} + \tilde{\mathbf{c}} \otimes \mathbf{b}) - (\alpha^2 T)^{-1}(\rho c)\tilde{\mathbf{c}} \otimes \tilde{\mathbf{c}}, \quad (3.89)$$

we may rewrite (3.88) in terms of $\tilde{\mathbf{P}}$ as

$$\det\{(\tilde{\mathbf{P}} - w\mathbf{1}) + \frac{i\omega}{\alpha}(\alpha_1\mathbf{b} \otimes \tilde{\mathbf{c}} + \tau\tilde{\mathbf{c}} \otimes \mathbf{b}) - (\alpha^2 T)^{-1}i\omega(s^2 k - \alpha_0\rho c)\tilde{\mathbf{c}} \otimes \tilde{\mathbf{c}}\} = 0. \quad (3.90)$$

Similarly to as seen Section 2.3.1, the secular equation (3.90) may be written in terms of definitions (2.110a) and (2.110b) as

$$\det\{(\tilde{\mathbf{P}} - w\mathbf{1}) + \frac{i\omega}{\alpha}(\alpha_1\beta\mathbf{n} \otimes \mathbf{n} + \tau\beta\mathbf{n} \otimes \mathbf{n}) - (\alpha^2 T)^{-1}i\omega(s^2 k - \alpha_0\rho c)\mathbf{n} \otimes \mathbf{n}\} = 0. \quad (3.91)$$

By simplification this equation becomes

$$\det\{(\tilde{\mathbf{P}} - w\mathbf{1}) + \frac{i\omega}{\alpha}\beta(\alpha_1 + \tau) - (\alpha^2 T)^{-1}i\omega(s^2 k - \alpha_0\rho c)\}\mathbf{n} \otimes \mathbf{n} = 0. \quad (3.92)$$

By using the standard identity (2.60) we get

$$\det(\tilde{\mathbf{P}} - w\mathbf{1}) + i\omega[\alpha^{-1}\beta(\alpha_1 + \tau) - (\alpha^2 T)^{-1}(s^2 k - \alpha_0 \rho c)]\mathbf{n} \cdot (\tilde{\mathbf{P}} - w\mathbf{1})^{\text{adj}}\mathbf{n} = 0. \quad (3.93)$$

This is the secular equation for anisotropic TRDTE+GTE (1) which is constrained by the usual deformation temperature constraint and has not previously appeared in the literature.

The tensor $\tilde{\mathbf{P}}$ might be written in terms of the isentropic tensor as follows

$$\tilde{\mathbf{P}} := \hat{\mathbf{Q}} - \frac{T}{\rho c}(\mathbf{b} + \frac{\rho c}{\alpha T}\tilde{\mathbf{c}}) \otimes (\mathbf{b} + \frac{\rho c}{\alpha T}\tilde{\mathbf{c}}). \quad (3.94)$$

To non-dimensionalise (3.94) using the dimensionless quantities (2.103), we will get the same equation as (2.104). Now the secular equation (3.93) can be written in terms of non-dimensional quantities as

$$\det(\tilde{\mathbf{P}}' - w'\mathbf{1}) + i\omega'\tilde{\sigma}[\varepsilon^{\frac{1}{2}}(\alpha'_1 + \tau') - \tilde{\sigma}(\frac{1}{w'} - \alpha'_0)]\mathbf{n} \cdot (\tilde{\mathbf{P}}' - w'\mathbf{1})^{\text{adj}}\mathbf{n} = 0. \quad (3.95)$$

This equation may be written in a clearer form as

$$w \det(w\mathbf{1} - \tilde{\mathbf{P}}) - i\omega\tilde{\sigma}[w\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau) - \tilde{\sigma}(1 - w\alpha_0)]\mathbf{n} \cdot (w\mathbf{1} - \tilde{\mathbf{P}})^{\text{adj}}\mathbf{n} = 0. \quad (3.96)$$

Now we need to group terms in w together

$$w \det(w\mathbf{1} - \tilde{\mathbf{P}}) - i\omega\tilde{\sigma}[w(\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau) + \alpha_0\tilde{\sigma}) - \tilde{\sigma}]\mathbf{n} \cdot (w\mathbf{1} - \tilde{\mathbf{P}})^{\text{adj}}\mathbf{n} = 0. \quad (3.97)$$

This equation may be written in more simplified form as

$$\tilde{F}(w) - i\omega\tilde{\sigma}[w(\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau) + \alpha_0\tilde{\sigma}) - \tilde{\sigma}]\tilde{G}(w) = 0, \quad (3.98)$$

where $\tilde{F}(w)$ and $\tilde{G}(w)$ are defined earlier in (2.116). On putting $\tau = 0$ in (3.98) we get the corresponding secular equation (2.115) of TRDTE, as expected.

Low frequency expansions

Again, similar to the previous sections when $\omega = 0$ the roots of the secular equation are the zeros of $\tilde{F}(w) : \tilde{p}_i, i = 0, 1, 2, 3$, defining $\tilde{p}_0 \equiv 0$. Taylor expansions take the form

$$w_i(\omega) = \tilde{p}_i + \sum_{n=1}^{\infty} d_n^{(i)}(-i\omega)^n, \quad i = 0, 1, 2, 3. \quad (3.99)$$

When $i = 0, n = 1$,

$$w_0(\omega) = \tilde{p}_0 + d_1^{(0)}(-i\omega) + O(\omega^2). \quad (3.100)$$

By substituting (3.100) into (3.98) we get

$$d_1^{(0)} = -\tilde{\sigma}^2 \frac{\tilde{W}_1 \tilde{W}_2}{\tilde{p}_1 \tilde{p}_2 \tilde{p}_3}. \quad (3.101)$$

The sign of $d_1^{(0)}$ depends on the sign of \tilde{p}_1 . Stability is satisfied when $d_1^{(0)} > 0$ and $d_1^{(0)}$ is positive if \tilde{p}_1 is negative. But if $\tilde{p}_1 > 0$ then $d_1^{(0)} < 0$ thus $w_0(\omega)$ is unstable. Inserting (3.101) into (3.100) we obtain

$$w_0(\omega) = i\omega \tilde{\sigma}^2 \frac{\tilde{G}(0)}{\tilde{F}'(0)} + O(\omega^2). \quad (3.102)$$

The exceptional case $\tilde{p}_1 = 0$ will be dealt with later. It represents a cross over for the branch $w_0(\omega)$ between instability for $\tilde{p}_1 > 0$ and stability for $\tilde{p}_1 < 0$.

When $i = 1, n = 1$

$$w_1(\omega) = \tilde{p}_1 + d_1^{(1)}(-i\omega) + O(\omega^2). \quad (3.103)$$

By substituting (3.103) into (3.98) we get

$$d_1^{(1)} = -\tilde{\sigma} [\tilde{p}_1 (\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau) + \alpha_0 \tilde{\sigma}) - \tilde{\sigma}] \frac{(\tilde{p}_1 - \tilde{W}_1)(\tilde{p}_1 - \tilde{W}_2)}{\tilde{p}_1(\tilde{p}_1 - \tilde{p}_2)(\tilde{p}_1 - \tilde{p}_3)}. \quad (3.104)$$

The sign of $d_1^{(1)}$ depends on the sign of $[\tilde{p}_1 (\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau) + \alpha_0 \tilde{\sigma}) - \tilde{\sigma}]$ and \tilde{p}_1 , stability being satisfied if $d_1^{(1)} > 0$, and $d_1^{(1)}$ is positive if

$$0 < \tilde{p}_1 < \frac{\tilde{\sigma}}{\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau) + \alpha_0 \tilde{\sigma}}.$$

If $\tilde{p}_1 < 0$ then $d_1^{(1)} < 0$, thus $w_1(\omega)$ is unstable. Thus,

$$w_1(\omega) = \tilde{p}_1 + i\omega \tilde{\sigma} [\tilde{p}_1 (\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau) + \alpha_0 \tilde{\sigma}) - \tilde{\sigma}] \frac{\tilde{G}(\tilde{p}_1)}{\tilde{F}'(\tilde{p}_1)} + O(\omega^2). \quad (3.105)$$

Similarly, when $i = 2, 3, n = 1$, we obtain

$$w_2(\omega) = \tilde{p}_2 + d_1^{(2)}(-i\omega) + O(\omega^2), \quad (3.105a)$$

$$w_3(\omega) = \tilde{p}_3 + d_1^{(3)}(-i\omega) + O(\omega^2). \quad (3.105b)$$

Substituting (3.105a) and (3.105b) into (3.98) we get

$$d_1^{(2)} = -\tilde{\sigma} [\tilde{p}_2(\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau) + \alpha_0\tilde{\sigma}) - \tilde{\sigma}] \frac{(\tilde{p}_2 - \tilde{W}_1)(\tilde{p}_2 - \tilde{W}_2)}{\tilde{p}_2(\tilde{p}_2 - \tilde{p}_1)(\tilde{p}_2 - \tilde{p}_3)}, \quad (3.105c)$$

$$d_1^{(3)} = -\tilde{\sigma} [\tilde{p}_3(\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau) + \alpha_0\tilde{\sigma}) - \tilde{\sigma}] \frac{(\tilde{p}_3 - \tilde{W}_1)(\tilde{p}_3 - \tilde{W}_2)}{\tilde{p}_3(\tilde{p}_3 - \tilde{p}_1)(\tilde{p}_3 - \tilde{p}_2)}. \quad (3.105d)$$

The signs of $d_1^{(2)}$ and $d_1^{(3)}$ depend only on the sign of $\tilde{p}_2(\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau) + \alpha_0\tilde{\sigma}) - \tilde{\sigma}$ and $\tilde{p}_3(\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau) + \alpha_0\tilde{\sigma}) - \tilde{\sigma}$, respectively; the sign of \tilde{p}_1 does not affect things here. Stability is satisfied if $d_1^{(2)}$ and $d_1^{(3)}$ are positive. So that is obtained if

$$\tilde{p}_2 < \frac{\tilde{\sigma}}{\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau) + \alpha_0\tilde{\sigma}},$$

and

$$\tilde{p}_3 < \frac{\tilde{\sigma}}{\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau) + \alpha_0\tilde{\sigma}},$$

respectively. Insert (3.105c) and (3.105d) into (3.105a) and (3.105b) we get

$$w_2(\omega) = \tilde{p}_2 + i\omega\tilde{\sigma} [\tilde{p}_2(\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau) + \alpha_0\tilde{\sigma}) - \tilde{\sigma}] \frac{\tilde{G}(\tilde{p}_2)}{\tilde{F}'(\tilde{p}_2)} + O(\omega^2), \quad (3.106)$$

$$w_3(\omega) = \tilde{p}_3 + i\omega\tilde{\sigma} [\tilde{p}_3(\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau) + \alpha_0\tilde{\sigma}) - \tilde{\sigma}] \frac{\tilde{G}(\tilde{p}_3)}{\tilde{F}'(\tilde{p}_3)} + O(\omega^2). \quad (3.107)$$

Summarising, if the quantity $\tilde{p}_i[\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau) + \alpha_0\tilde{\sigma}] - \tilde{\sigma}$ is negative for any $i = 1, 2, 3$, then the corresponding branch $w_i(\omega)$ is stable. Conversely, if this quantity is positive the corresponding branch is unstable.

High frequency expansions

The roots of the secular equation (3.98) in the high frequency case are given by the zeros of

$$[w(\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau) + \alpha_0\tilde{\sigma}) - \tilde{\sigma}] \tilde{G}(w) = 0.$$

So there are four roots, three of them are finite

$$w_1 = \tilde{W}_1, w_2 = \tilde{W}_2 \text{ and } w_3 = \tilde{W}_3 \equiv \frac{\tilde{\sigma}}{\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau) + \alpha_0\tilde{\sigma}},$$

and one is infinite. Power series expansions take the form

$$w_i(\omega) = \tilde{W}_i + \sum_{n=1}^{\infty} d_n^{(i)} (-i\omega)^{-n}, \quad i = 1, 2, 3. \quad (3.108)$$

When $i = 1, n = 1$, we obtain

$$w_1(\omega) = \tilde{W}_1 + d_1^{(1)}(-i\omega)^{-1} + O(\omega^{-2}). \quad (3.109)$$

Substituting (3.109) into (3.98) we obtain

$$d_1^{(1)} = \frac{\tilde{W}_1(\tilde{W}_1 - \tilde{p}_1)(\tilde{W}_1 - \tilde{p}_2)(\tilde{W}_1 - \tilde{p}_3)}{\tilde{\sigma}[\tilde{W}_1(\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau) + \alpha_0\tilde{\sigma}) - \tilde{\sigma}](\tilde{W}_1 - \tilde{W}_2)}. \quad (3.110)$$

The sign of $d_1^{(1)}$ depends only on the sign of $\tilde{W}_1(\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau) + \alpha_0\tilde{\sigma}) - \tilde{\sigma}$; the sign of \tilde{p}_1 does not affect things here. Stability is satisfied if $d_1^{(1)} < 0$, and $d_1^{(1)}$ is negative if

$$\tilde{W}_1 > \frac{\tilde{\sigma}}{\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau) + \alpha_0\tilde{\sigma}}.$$

Thus,

$$w_1(\omega) = \tilde{W}_1 + \frac{i\omega^{-1}\tilde{F}(\tilde{W}_1)}{\tilde{\sigma}[\tilde{W}_1(\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau) + \alpha_0\tilde{\sigma}) - \tilde{\sigma}]\tilde{G}'(\tilde{W}_1)} + O(\omega^{-2}). \quad (3.111)$$

Similarly, when $i = 2, n = 1$, we get

$$w_2(\omega) = \tilde{W}_2 + \frac{i\omega^{-1}\tilde{F}(\tilde{W}_2)}{\tilde{\sigma}[\tilde{W}_2(\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau) + \alpha_0\tilde{\sigma}) - \tilde{\sigma}]\tilde{G}'(\tilde{W}_2)} + O(\omega^{-2}). \quad (3.112)$$

$w_2(\omega)$ is stable if

$$\tilde{W}_2 > \frac{\tilde{\sigma}}{\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau) + \alpha_0\tilde{\sigma}}.$$

When $i = 3, n = 1$

$$w_3(\omega) = \tilde{W}_3 + d_1^{(3)}(-i\omega)^{-1} + O(\omega^{-2}), \quad (3.113)$$

where

$$\tilde{W}_3 = \frac{\tilde{\sigma}}{\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau) + \alpha_0\tilde{\sigma}}.$$

By substituting (3.113) into (3.98) we get

$$d_1^{(3)} = \frac{-\tilde{F}(\tilde{W}_3)}{\tilde{\sigma}(\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau) + \alpha_0\tilde{\sigma})\tilde{G}(\tilde{W}_3)}, \quad (3.114)$$

then,

$$d_1^{(3)} = \frac{-\tilde{W}_3(\tilde{W}_3 - \tilde{p}_1)(\tilde{W}_3 - \tilde{p}_2)(\tilde{W}_3 - \tilde{p}_3)}{\tilde{\sigma}(\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau) + \alpha_0\tilde{\sigma})(\tilde{W}_3 - \tilde{W}_1)(\tilde{W}_3 - \tilde{W}_2)}. \quad (3.115)$$

By simplifying we get

$$d_1^{(3)} = \frac{-(\tilde{W}_3 - \tilde{p}_1)(\tilde{W}_3 - \tilde{p}_2)(\tilde{W}_3 - \tilde{p}_3)}{(\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau) + \alpha_0 \tilde{\sigma})^2 (\tilde{W}_3 - \tilde{W}_1)(\tilde{W}_3 - \tilde{W}_2)}. \quad (3.116)$$

The sign of $d_1^{(3)}$ may be determined in two special cases; one as $\tilde{\sigma} \rightarrow 0$, a purely thermal constraint, and the other as $\tilde{\sigma} \rightarrow \infty$, a purely mechanical constraint. So, as $\tilde{\sigma} \rightarrow 0$, $\tilde{W}_3 \rightarrow 0$, then

$$d_1^{(3)} = \frac{\tilde{p}_1 \tilde{p}_2 \tilde{p}_3}{(\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau))^2 \tilde{W}_1 \tilde{W}_2}.$$

Thus, it is clear that $w_3(\omega)$ is stable if $\tilde{p}_1 < 0$, but if $\tilde{p}_1 > 0$ we find that $w_3(\omega)$ is unstable. On the other hand, as $\tilde{\sigma} \rightarrow \infty$ we find that $\tilde{W}_3 \rightarrow \alpha_0^{-1}$ and $d_1^{(3)} \rightarrow 0$, and high powers are needed in the expansion (3.113).

The fourth root may be written as

$$w_4(\omega) = i\omega A_1 + B_1 + O(\omega^{-1}). \quad (3.117)$$

Substituting (3.117) into (3.98) we get

$$\begin{aligned} & (i\omega A_1 + B_1)(i\omega A_1 + B_1 - \tilde{p}_1)(i\omega A_1 + B_1 - \tilde{p}_2)(i\omega A_1 + B_1 - \tilde{p}_3) \\ & - i\omega \tilde{\sigma} \left\{ [(i\omega A_1 + B_1)(\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau) + \alpha_0 \tilde{\sigma}) - \tilde{\sigma}](i\omega A_1 + B_1 - \tilde{W}_1)(i\omega A_1 + B_1 - \tilde{W}_2) \right\} = 0. \end{aligned} \quad (3.118)$$

Multiplying this equation by $(i\omega)^{-4}$ we get

$$\begin{aligned} & (A_1 + B_1(i\omega)^{-1})(A_1 + (B_1 - \tilde{p}_1)(i\omega)^{-1})(A_1 + (B_1 - \tilde{p}_2)(i\omega)^{-1})(A_1 + (B_1 - \tilde{p}_3)(i\omega)^{-1}) \\ & - \tilde{\sigma} \left\{ [(A_1 + B_1(i\omega)^{-1})(\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau) + \alpha_0 \tilde{\sigma}) - \tilde{\sigma}(i\omega)^{-1}](A_1 + (B_1 - \tilde{W}_1)(i\omega)^{-1}) \right. \\ & \left. (A_1 + (B_1 - \tilde{W}_2)(i\omega)^{-1}) \right\} = 0. \end{aligned} \quad (3.119)$$

For large ω we obtain

$$A_1^4 - \tilde{\sigma} A_1^3 (\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau) + \alpha_0 \tilde{\sigma}) = 0.$$

So

$$A_1 = 0, 0, 0, \tilde{\sigma}(\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau) + \alpha_0 \tilde{\sigma}). \quad (3.120)$$

The three zero roots correspond to the roots already found. Expanding (3.119):

$$\begin{aligned} & A_1^3 \{ A_1 + [(B_1 - \tilde{p}_3)(i\omega)^{-1} + (B_1 - \tilde{p}_2)(i\omega)^{-1} + (B_1 - \tilde{p}_1)(i\omega)^{-1} + B_1(i\omega)^{-1}] \} \\ & - A_1^2 \tilde{\sigma} \{ (A_1 + B_1(i\omega)^{-1})(\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau) + \alpha_0 \tilde{\sigma}) - \tilde{\sigma}(i\omega)^{-1} + (B_1 - \tilde{W}_2)(\alpha_1 \varepsilon^{\frac{1}{2}} + \alpha_0 \tilde{\sigma})(i\omega)^{-1} \\ & \quad + (B_1 - \tilde{W}_1)(\alpha_1 \varepsilon^{\frac{1}{2}} + \alpha_0 \tilde{\sigma})(i\omega)^{-1} \} = 0. \end{aligned} \quad (3.121)$$

Cancel A_1^2 , $A_1 \neq 0$,

$$\begin{aligned} & A_1 \{ A_1 + [(B_1 - \tilde{p}_3)(i\omega)^{-1} + (B_1 - \tilde{p}_2)(i\omega)^{-1} + (B_1 - \tilde{p}_1)(i\omega)^{-1} + B_1(i\omega)^{-1}] \} \\ & + \tilde{\sigma} \{ (A_1 + B_1(i\omega)^{-1})(\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau) + \alpha_0 \tilde{\sigma}) - \tilde{\sigma}(i\omega)^{-1} + (B_1 - \tilde{W}_2)(\alpha_1 \varepsilon^{\frac{1}{2}} + \alpha_0 \tilde{\sigma})(i\omega)^{-1} \\ & \quad + (B_1 - \tilde{W}_1)(\alpha_1 \varepsilon^{\frac{1}{2}} + \alpha_0 \tilde{\sigma})(i\omega)^{-1} \} = 0 \end{aligned} \quad (3.122)$$

Coefficients of $(i\omega)^{-1}$ are

$$A_1 [4B_1 - (\tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3)] + \tilde{\sigma}(\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau) + \alpha_0 \tilde{\sigma}) \{ 3B_1 - (\tilde{W}_1 + \tilde{W}_2) \} - \tilde{\sigma}^2 = 0. \quad (3.123)$$

Using (3.116) we get B_1 :

$$B_1 = (\tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3) - (\tilde{W}_1 + \tilde{W}_2) - \frac{\tilde{\sigma}}{\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau) + \alpha_0 \tilde{\sigma}}.$$

Inserting A_1 and B_1 into (3.117), thus

$$w_4(\omega) = i\omega \tilde{\sigma} (\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau) + \alpha_0 \tilde{\sigma}) + (\tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3) - (\tilde{W}_1 + \tilde{W}_2) - \frac{\tilde{\sigma}}{\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau) + \alpha_0 \tilde{\sigma}} + O(\omega^{-1}). \quad (3.124)$$

It is clear that $w_4(\omega)$ is unstable because $\text{Im } w_4(\omega) > 0$.

Summarising, if the quantity $\tilde{W}_i[\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau) + \alpha_0 \tilde{\sigma}] - \tilde{\sigma}$ is positive for any $i = 1, 2$, then the corresponding branch $w_i(\omega)$, $i = 1, 2$, is stable in the high frequency. Conversely, if this quantity is negative the corresponding branch is unstable. The branch $w_3(\omega)$ is stable if $\tilde{p}_1 < 0$, unstable if $\tilde{p}_1 > 0$ when $\tilde{\sigma} \rightarrow 0$. The branch $w_4(\omega)$ always unstable.

From the secular equation (3.98), and previous arguments, we see that a branch cannot change from stable to unstable, or vice versa, for intermediate frequencies $0 < \omega < \infty$. This is borne out by Figures 3.6–3.8.

The exceptional case $\tilde{p}_1 = 0$.

When $\tilde{p}_1 = 0$ the secular equation (3.98), with the aid of (2.116), becomes

$$w^2(w - \tilde{p}_2)(w - \tilde{p}_3) - i\omega\tilde{\sigma} [w(\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau) + \alpha_0\tilde{\sigma}) - \tilde{\sigma}](w - \tilde{W}_1)(w - \tilde{W}_2) = 0.$$

For low frequencies we try the balance $w = A(-i\omega)^n$ and substitute into this secular equation in order to determine n . We find that $n = 1/2$ and then w is given by

$$w = \pm \left(-i\omega\tilde{\sigma}^2 \frac{\tilde{W}_1\tilde{W}_2}{\tilde{p}_2\tilde{p}_3} \right)^{\frac{1}{2}}.$$

These two branches begin at the origin and have arguments $-\pi/4$ and $3\pi/4$ in the complex w plane. This can be seen in Figure 3.8.

Numerical results

In Figure 3.6 we have taken $\tilde{p}_1 > 0$. The branch $w_0(\omega)$ beginning at the origin is unstable in each part of the Figure. All the other branches begin to the right of this branch. If α_0 and α_1 are small enough then

$$\tilde{p}_i(\varepsilon^{\frac{1}{2}}(\alpha_1 + \tau) + \alpha_0\tilde{\sigma}) - \tilde{\sigma} < 0, \quad \text{for } i = 1, 2, 3,$$

and so all the branches $w_i(\omega)$, $i = 1, 2, 3$, are stable in the low frequency. This can be seen in the first subfigure (a) of Figure 3.6 where α_0 and α_1 are small. As α_0 and α_1 increase, first $w_3(\omega)$ becomes unstable, see part (b), and as they increase further other branches become unstable.

In Figure 3.7 we have taken $\tilde{p}_1 < 0$. The branch $w_1(\omega)$ beginning at $w = \tilde{p}_1$ is unstable in each part of the Figure. All the other branches begin to the right of this branch. As in Figure 3.6, increasing α_0 and α_1 leads to increasing instability but $w_0(\omega)$ maintains the stability in each part of the Figure.

In Figure 3.8 we illustrate the exceptional case $\tilde{p}_1 = 0$. Now two branches emanate from the origin, namely, $w_0(\omega)$ and $w_1(\omega)$, one stable and the other unstable, one with argument $-\pi/4$ and the other with argument $3\pi/4$. The same increasing instability with increasing α_0 and α_1 is observed. As in the Figure 3.7 the branch $w_0(\omega)$ maintains the stability in each part of the Figure.

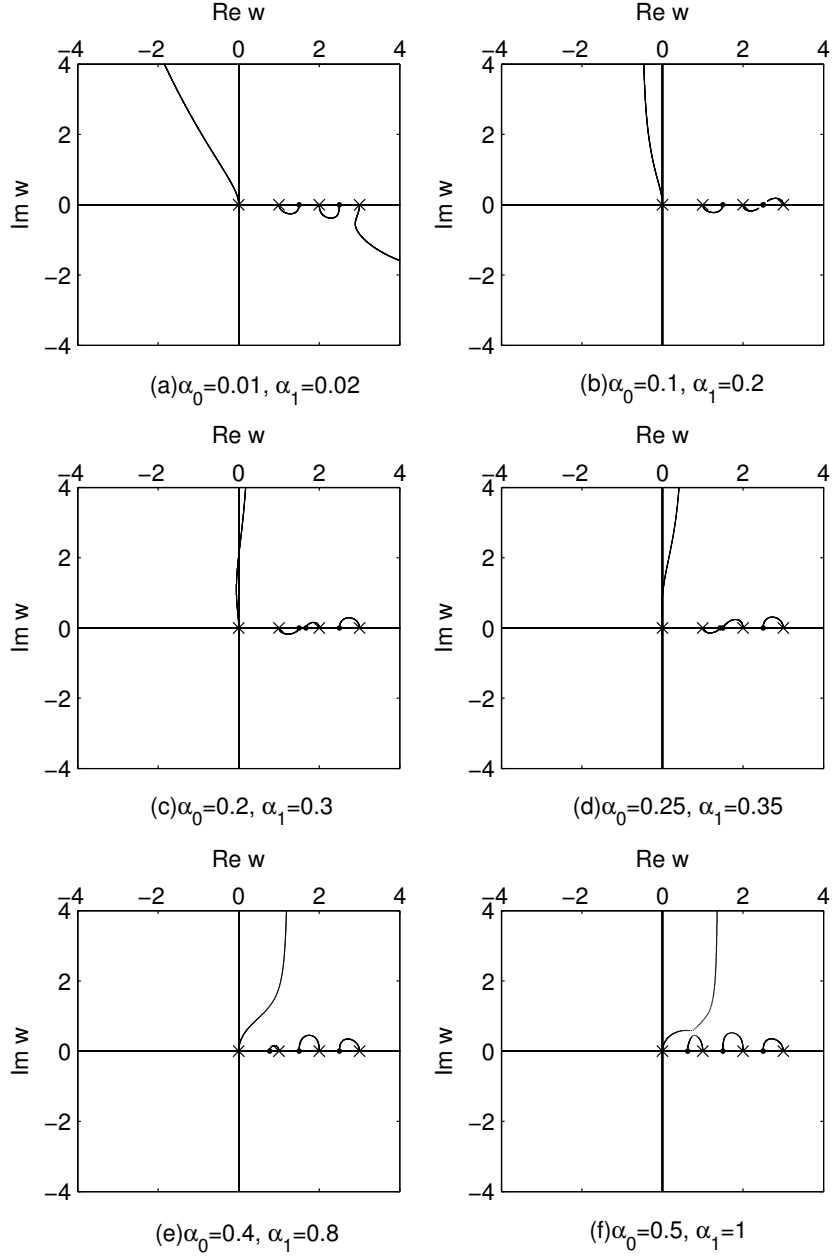


Figure 3.6: The longitudinal squared wave speeds of constrained anisotropic TRDTE+GTE (1). For each part, $\tilde{p}_1 = 1, \tilde{p}_2 = 2, \tilde{p}_3 = 3, \tilde{W}_1 = 1.5, \tilde{W}_2 = 2.5, \tilde{\sigma} = 1, \varepsilon = 1, \tau = 0.1$

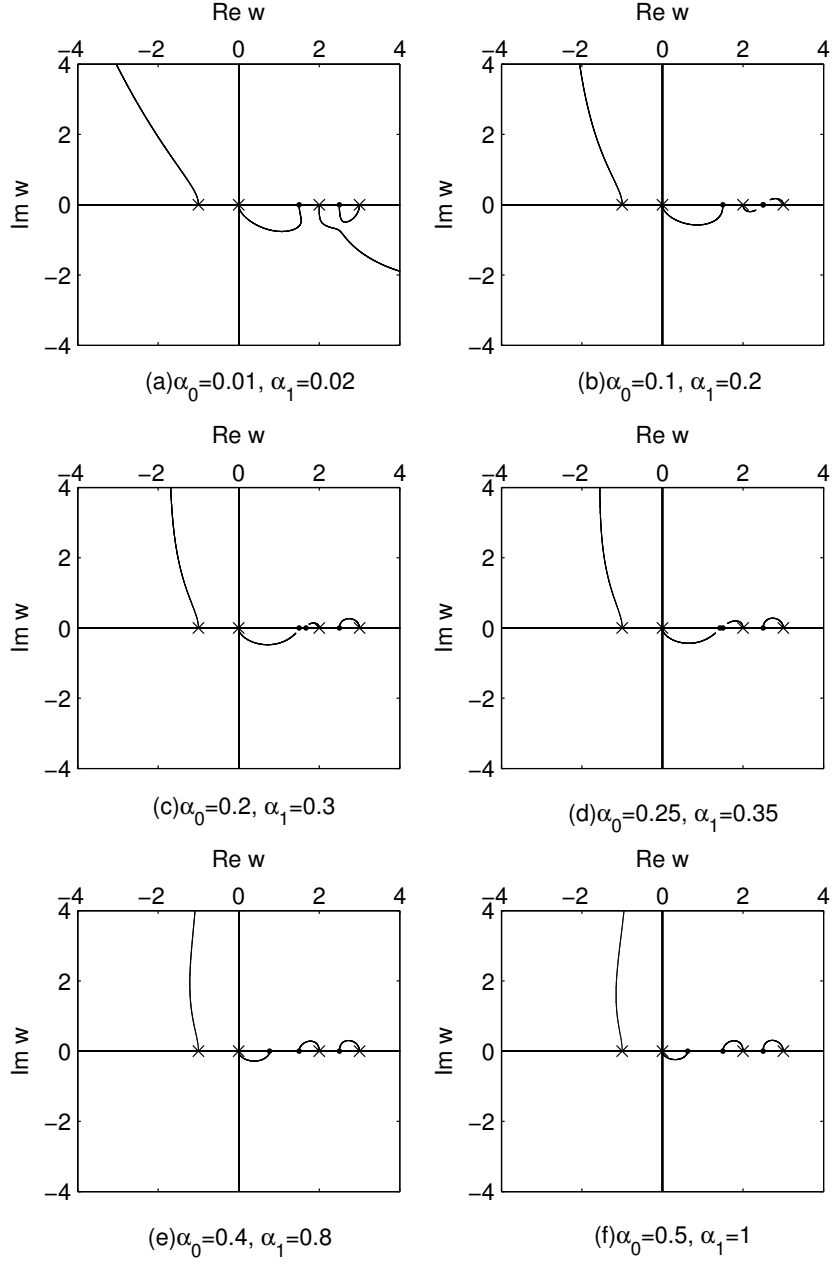


Figure 3.7: The longitudinal squared wave speeds of constrained anisotropic TRDTE+GTE (1). For each part, $\tilde{p}_1 = -1, \tilde{p}_2 = 2, \tilde{p}_3 = 3, \tilde{W}_1 = 1.5, \tilde{W}_2 = 2.5, \tilde{\sigma} = 1, \varepsilon = 1, \tau = 0.1$

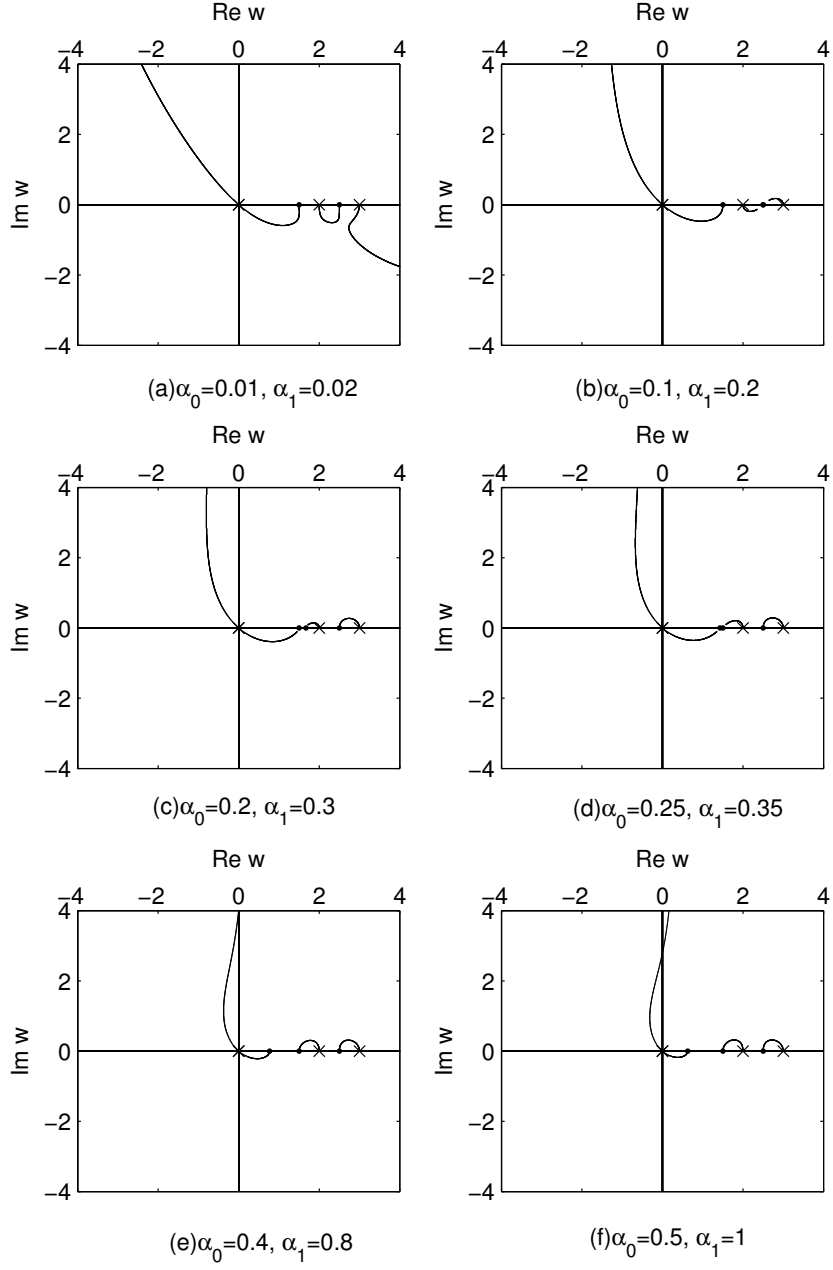


Figure 3.8: The longitudinal squared wave speeds of constrained anisotropic TRDTE+GTE (1). For each part, $\tilde{p}_1 = 0, \tilde{p}_2 = 2, \tilde{p}_3 = 3, \tilde{W}_1 = 1.5, \tilde{W}_2 = 2.5, \tilde{\sigma} = 1, \varepsilon = 1, \tau = 0.1$

3.3.2 Alternative form of deformation-temperature constraints

In this section we will use equations (3.81)_{1,2} and (2.144),

$$\left. \begin{aligned} \tilde{c}_{ijkl}u_{k,jl} - \beta_{ij}(\theta_{,j} + \alpha_1\dot{\theta}_{,j}) + \tilde{N}_{ij}\tilde{\eta}_{,j} &= \rho\ddot{u}_i, \\ k_{ij}\theta_{,ij} - T\beta_{pq}\left(1 + \tau\frac{\partial}{\partial t}\right)\dot{u}_{p,q} - \rho c(\dot{\theta} + \alpha_0\ddot{\theta}) - T\alpha\dot{\tilde{\eta}} &= 0, \\ \tilde{N}_{pq}u_{p,q} - \alpha(\theta + \alpha_0\dot{\theta}) &= 0. \end{aligned} \right\} \quad (3.125)$$

The secular equation

Now we follow the same steps as Section 2.3.2 to get the secular equation. Firstly, looking for solution for equations (3.125) in the form of the plane harmonic waves (2.86) by inserting (2.86) into (3.125), we get the same equations we had before: (3.82), (3.83) and (2.146)₃,

$$\left. \begin{aligned} (\tilde{Q}_{ip} - \rho s^{-2}\delta_{ip})U_p + i(\omega s)^{-1}[b_i(1 - i\omega\alpha_1)\Theta - \tilde{c}_i\tilde{H}] &= 0, \\ \omega s T b_p(1 - i\omega\tau)U_p - i\alpha T \tilde{H} + (\omega s^2 k - i\rho c(1 - i\omega\alpha_0))\Theta &= 0, \\ i\omega s \tilde{c}_p U_p - \alpha(1 - i\omega\alpha_0)\Theta &= 0, \end{aligned} \right\} \quad (3.126)$$

where $\tilde{N}_{pq}n_q = \tilde{c}_p$, $\beta_{ij}n_j = b_i$, $k_{ij}n_j = k$. Eliminate Θ and \tilde{H} between (3.126). From (3.126)₃ we find that

$$\Theta = \frac{i\omega s \tilde{c}_p U_p}{\alpha(1 - i\omega\alpha_0)}. \quad (3.127)$$

By substituting (3.127) into (3.126)₂ we get

$$\tilde{H} = -i\alpha^{-1}\omega s b_p(1 - i\omega\tau)U_p + (\alpha^2 T)^{-1}(1 - i\omega\alpha_0)^{-1}\omega s \tilde{c}_p U_p \left(\omega s^2 k - i\rho c(1 - i\omega\alpha_0) \right). \quad (3.128)$$

Insert (3.127) and (3.128) into (3.126)₁ we obtain

$$\left\{ (\tilde{Q}_{ip} - \rho s^{-2}\delta_{ip}) - \alpha^{-1}(1 - i\omega\alpha_0)^{-1} \left(b_i \tilde{c}_p(1 - i\omega\alpha_1) + \tilde{c}_i b_p(1 - i\omega\alpha_0)(1 - i\omega\tau) \right) - i(\alpha^2 T)^{-1} (1 - i\omega\alpha_0)^{-1}(\omega s^2 k - i\rho c(1 - i\omega\alpha_0))\tilde{c}_i \tilde{c}_p \right\} U_p = 0. \quad (3.129)$$

Expanding this equation we get

$$\left\{ \tilde{Q}_{ip} - \alpha^{-1}(1 - i\omega\alpha_0)^{-1} \left[(b_i \tilde{c}_p + \tilde{c}_i b_p) - i\omega(\alpha_1 b_i \tilde{c}_p + (\alpha_0 + \tau)\tilde{c}_i b_p - i\omega\alpha_0 \tau \tilde{c}_i b_p) \right] - i(\alpha^2 T)^{-1}(1 - i\omega\alpha_0)^{-1}(\omega s^2 k - i\rho c - \rho c \omega \alpha_0)\tilde{c}_i \tilde{c}_p - \rho s^{-2}\delta_{ip} \right\} U_p = 0. \quad (3.130)$$

Rearranging this equation we get

$$\left\{ \tilde{Q}_{ip} - (1 - i\omega\alpha_0)^{-1} \left[\alpha^{-1} (b_i \tilde{c}_p + \tilde{c}_i b_p) + (\alpha^2 T)^{-1} \rho c \tilde{c}_i \tilde{c}_p \right] + i\omega (1 - i\omega\alpha_0)^{-1} \left[\alpha^{-1} \left(\alpha_1 b_i \tilde{c}_p + (\alpha_0 + \tau) \tilde{c}_i b_p - i\omega\alpha_0 \tau \tilde{c}_i b_p \right) - (\alpha^2 T)^{-1} (s^2 k - \rho c \alpha_0) \tilde{c}_p \tilde{c}_i \right] - \rho s^{-2} \delta_{ip} \right\} U_p = 0.$$

The non-zero amplitudes U_p satisfy (3.126) if and only if

$$\det \left\{ \tilde{\mathbf{Q}} - (1 - i\omega\alpha_0)^{-1} \left[\alpha^{-1} (\mathbf{b} \otimes \tilde{\mathbf{c}} + \tilde{\mathbf{c}} \otimes \mathbf{b}) + (\alpha^2 T)^{-1} \rho c \tilde{\mathbf{c}} \otimes \tilde{\mathbf{c}} \right] + i\omega (1 - i\omega\alpha_0)^{-1} \left[\alpha^{-1} (\alpha_1 \mathbf{b} \otimes \tilde{\mathbf{c}} + (\alpha_0 + \tau - i\omega\alpha_0 \tau) \tilde{\mathbf{c}} \otimes \mathbf{b}) - (\alpha^2 T)^{-1} (s^2 k - \rho c \alpha_0) \tilde{\mathbf{c}} \otimes \tilde{\mathbf{c}} \right] - w \mathbf{1} \right\} = 0. \quad (3.131)$$

By defining

$$\tilde{\mathbf{S}} := \tilde{\mathbf{Q}} - (1 - i\omega\alpha_0)^{-1} \left[\alpha^{-1} (\mathbf{b} \otimes \tilde{\mathbf{c}} + \tilde{\mathbf{c}} \otimes \mathbf{b}) + \frac{\rho c}{\alpha^2 T} \tilde{\mathbf{c}} \otimes \tilde{\mathbf{c}} \right] \quad (3.132)$$

equation (3.131) may be written as

$$\det \left\{ (\tilde{\mathbf{S}} - w \mathbf{1}) + \frac{i\omega}{(1 - i\omega\alpha_0)} \left[\frac{\alpha_1}{\alpha} \mathbf{b} \otimes \tilde{\mathbf{c}} + \left(\frac{\alpha_0 + \tau - i\omega\alpha_0 \tau}{\alpha} \right) \tilde{\mathbf{c}} \otimes \mathbf{b} - \frac{(s^2 k - \rho c \alpha_0)}{\alpha^2 T} \tilde{\mathbf{c}} \otimes \tilde{\mathbf{c}} \right] \right\} = 0. \quad (3.133)$$

Now we need to rewrite (3.132) in terms of the definitions (2.110a) and (2.110b) we will get the same equation as (2.154). The dimensionless form of (3.132) will be similar to (2.155). Also in terms of the definitions (2.110a) and (2.110b) the secular equation (3.133) may be written as

$$\det \left\{ (\tilde{\mathbf{S}} - w \mathbf{1}) + \frac{i\omega}{(1 - i\omega\alpha_0)} \left[\alpha^{-1} \beta (\alpha_1 + \alpha_0 + \tau (1 - i\omega\alpha_0)) - \frac{(s^2 k - \rho c \alpha_0)}{\alpha^2 T} \right] \mathbf{n} \otimes \mathbf{n} \right\} = 0. \quad (3.134)$$

By using the standard identity (2.60), equation (3.134) may be rewritten as

$$\det(\tilde{\mathbf{S}} - w \mathbf{1}) + \frac{i\omega}{(1 - i\omega\alpha_0)} \left[\alpha^{-1} \beta (\alpha_1 + \alpha_0 + \tau (1 - i\omega\alpha_0)) - \frac{(s^2 k - \rho c \alpha_0)}{\alpha^2 T} \right] \mathbf{n} \cdot (\tilde{\mathbf{S}} - w \mathbf{1})^{\text{adj}} \mathbf{n} = 0. \quad (3.135)$$

To non-dimensionalise this equation we use the dimensionless quantities (2.103), to get

$$\det(\tilde{\mathbf{S}} - w \mathbf{1}) + \frac{i\omega \tilde{\sigma}}{(1 - i\omega\alpha_0)} \left[\varepsilon^{\frac{1}{2}} (\alpha_1 + \alpha_0 + \tau (1 - i\omega\alpha_0)) - \tilde{\sigma} (w^{-1} - \alpha_0) \right] \mathbf{n} \cdot (\tilde{\mathbf{S}} - w \mathbf{1})^{\text{adj}} \mathbf{n} = 0. \quad (3.136)$$

This equation may be written as

$$w \det(w\mathbf{1} - \tilde{\mathbf{S}}) - \frac{i\omega\tilde{\sigma}}{(1 - i\omega\alpha_0)} \left[w \left(\varepsilon^{\frac{1}{2}} (\alpha_1 + \alpha_0 + \tau(1 - i\omega\alpha_0)) + \alpha_0\tilde{\sigma} \right) - \tilde{\sigma} \right] \mathbf{n} \cdot (w\mathbf{1} - \tilde{\mathbf{S}})^{\text{adj}} \mathbf{n} = 0. \quad (3.137)$$

Now we want to rewrite the secular equation in terms of $\tilde{\mathbf{P}}$, defined by (2.112). We can see that the first term and the last term of equation (3.137) are similar to those of (2.162). Thus the secular equation (3.137) is now written as

$$w \det(w\mathbf{1} - \tilde{\mathbf{P}}) + \frac{i\omega\alpha_0\tilde{\sigma}w}{(1 - i\omega\alpha_0)} (2\varepsilon^{\frac{1}{2}} + \tilde{\sigma}) \mathbf{n} \cdot (w\mathbf{1} - \tilde{\mathbf{P}})^{\text{adj}} \mathbf{n} - \frac{i\omega\tilde{\sigma}}{(1 - i\omega\alpha_0)} \left[w \left(\varepsilon^{\frac{1}{2}} (\alpha_1 + \alpha_0 + \tau(1 - i\omega\alpha_0)) + \alpha_0\tilde{\sigma} \right) - \tilde{\sigma} \right] \mathbf{n} \cdot (w\mathbf{1} - \tilde{\mathbf{P}})^{\text{adj}} \mathbf{n} = 0. \quad (3.138)$$

Simplifying this equation we get

$$w \det(w\mathbf{1} - \tilde{\mathbf{P}}) - \frac{i\omega\tilde{\sigma}}{(1 - i\omega\alpha_0)} \left[w\varepsilon^{\frac{1}{2}} \left((\alpha_1 - \alpha_0) + \tau(1 - i\omega\alpha_0) \right) - \tilde{\sigma} \right] \mathbf{n} \cdot (w\mathbf{1} - \tilde{\mathbf{P}})^{\text{adj}} \mathbf{n} = 0. \quad (3.139)$$

This is the secular equation for anisotropic TRDTE+GTE (1) which is constrained by the alternative deformation temperature constraint and has not previously appeared in the literature.

We want to write equation (3.139) in terms of the eigenvalues $\tilde{p}_1, \tilde{p}_2, \tilde{p}_3$ as

$$w(w - \tilde{p}_1)(w - \tilde{p}_2)(w - \tilde{p}_3) - \frac{i\omega\tilde{\sigma}}{(1 - i\omega\alpha_0)} \left[w\varepsilon^{\frac{1}{2}} \left((\alpha_1 - \alpha_0) + \tau(1 - i\omega\alpha_0) \right) - \tilde{\sigma} \right] \left[n_1^2(w - \tilde{p}_2)(w - \tilde{p}_3) + n_2^2(w - \tilde{p}_1)(w - \tilde{p}_3) + n_3^2(w - \tilde{p}_1)(w - \tilde{p}_2) \right] = 0. \quad (3.140)$$

Similarly to (2.167), the quadratic part within square brackets has zeros at $w = \tilde{W}_i$, where $i = 1, 2$, for which

$$\tilde{p}_1 < \tilde{W}_1 < \tilde{p}_2 < \tilde{W}_2 < \tilde{p}_3, \quad (3.140a)$$

so that equation (3.140) may be written as

$$\tilde{F}(w) - \frac{i\omega\tilde{\sigma}}{(1 - i\omega\alpha_0)} \left[w\varepsilon^{\frac{1}{2}} \left((\alpha_1 - \alpha_0) + \tau(1 - i\omega\alpha_0) \right) - \tilde{\sigma} \right] \tilde{G}(w) = 0, \quad (3.141)$$

where $\tilde{F}(w)$ and $\tilde{G}(w)$ are defined earlier in (2.116).

Low frequency expansions

When $\omega \rightarrow 0$ the roots of the secular equation (3.141) are the zeros of $\tilde{F}(w) : \tilde{p}_i, i = 0, 1, 2, 3$, defining $\tilde{p}_0 \equiv 0$. Taylor expansions in this case take the form

$$w_i(\omega) = \tilde{p}_i + \sum_{n=1}^{\infty} d_n^{(i)} (-i\omega)^n, \quad i = 0, 1, 2, 3. \quad (3.142)$$

When $i = 0, n = 1$ we get

$$w_0(\omega) = \tilde{p}_0 + d_1^{(0)} (-i\omega) + O(\omega^2). \quad (3.143)$$

Substituting (3.143) into (3.141) we obtain

$$d_1^{(0)} = -\tilde{\sigma}^2 \frac{\tilde{W}_1 \tilde{W}_2}{\tilde{p}_1 \tilde{p}_2 \tilde{p}_3}. \quad (3.144)$$

The sign of $d_1^{(0)}$ depends on the sign of \tilde{p}_1 . Stability is satisfied when $d_1^{(0)} > 0$, and $d_1^{(0)}$ is positive if \tilde{p}_1 is negative, so if $\tilde{p}_1 > 0$ then $w_0(\omega)$ is unstable.

Substituting (3.144) into (3.143) to get

$$w_0(\omega) = i\omega \tilde{\sigma} \frac{\tilde{G}(0)}{\tilde{F}'(0)} + O(\omega^2). \quad (3.145)$$

The exceptional case \tilde{p}_0 will be dealt with later.

When $i = 1, n = 1$ we get

$$w_1(\omega) = \tilde{p}_1 + d_1^{(1)} (-i\omega) + O(\omega^2). \quad (3.146)$$

Substituting (3.146) into (3.141) we obtain

$$d_1^{(1)} = -\tilde{\sigma} \left[\tilde{p}_1 \varepsilon^{\frac{1}{2}} (\alpha_1 - \alpha_0 + \tau) - \tilde{\sigma} \right] \frac{(\tilde{p}_1 - \tilde{W}_1)(\tilde{p}_1 - \tilde{W}_2)}{\tilde{p}_1 (\tilde{p}_1 - \tilde{p}_2)(\tilde{p}_1 - \tilde{p}_3)}. \quad (3.147)$$

The sign of $d_1^{(1)}$ depends on the sign of $[\tilde{p}_1 \varepsilon^{\frac{1}{2}} (\alpha_1 - \alpha_0 + \tau) - \tilde{\sigma}]$ and \tilde{p}_1 . From (3.146) it is clear that stability is satisfied if $d_1^{(1)} > 0$, and $d_1^{(1)}$ is positive if $\tilde{p}_1 > 0$ and

$$\begin{aligned} & \tilde{p}_1 \varepsilon^{\frac{1}{2}} (\alpha_1 - \alpha_0 + \tau) - \tilde{\sigma} < 0, \\ \Rightarrow \quad & \tilde{p}_1 < \frac{\tilde{\sigma}}{\varepsilon^{\frac{1}{2}} (\alpha_1 - \alpha_0 + \tau)}. \end{aligned}$$

Again, from (3.147) if $\tilde{p}_1 < 0$, we find that $d_1^{(1)} < 0$ then $w_1(\omega)$ is unstable. The exceptional case \tilde{p}_0 is considered later.

Substituting (3.147) into (3.146) we get

$$w_1(\omega) = \tilde{p}_1 + i\omega\tilde{\sigma}\left[\tilde{p}_1\varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0 + \tau) - \tilde{\sigma}\right]\frac{\tilde{G}(\tilde{p}_1)}{\tilde{F}'(\tilde{p}_1)} + O(\omega^2). \quad (3.148)$$

Similarly, when $i = 2, 3$ we obtain

$$w_2(\omega) = \tilde{p}_2 + d_1^{(2)}(-i\omega) + O(\omega^2), \quad (3.149)$$

$$w_3(\omega) = \tilde{p}_3 + d_1^{(3)}(-i\omega) + O(\omega^2). \quad (3.150)$$

Substituting (3.149) and (3.150) into (3.141) we obtain

$$d_1^{(2)} = -\tilde{\sigma}\left[\tilde{p}_2\varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0 + \tau) - \tilde{\sigma}\right]\frac{(\tilde{p}_2 - \tilde{W}_1)(\tilde{p}_2 - \tilde{W}_2)}{\tilde{p}_2(\tilde{p}_2 - \tilde{p}_1)(\tilde{p}_2 - \tilde{p}_3)}, \quad (3.151)$$

$$d_1^{(3)} = -\tilde{\sigma}\left[\tilde{p}_3\varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0 + \tau) - \tilde{\sigma}\right]\frac{(\tilde{p}_3 - \tilde{W}_1)(\tilde{p}_3 - \tilde{W}_2)}{\tilde{p}_3(\tilde{p}_3 - \tilde{p}_1)(\tilde{p}_3 - \tilde{p}_2)}. \quad (3.152)$$

The sign of $d_1^{(2)}$ and $d_1^{(3)}$ depend only on the sign of $\left[\tilde{p}_2\varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0 + \tau) - \tilde{\sigma}\right]$ and $\left[\tilde{p}_3\varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0 + \tau) - \tilde{\sigma}\right]$; respectively, and the sign of \tilde{p}_1 here does not affect things. It is clear that stability is satisfied if $d_1^{(2)}$ and $d_1^{(3)}$ are positive and this obtained by

$$\tilde{p}_2 < \frac{\tilde{\sigma}}{\varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0 + \tau)},$$

and

$$\tilde{p}_3 < \frac{\tilde{\sigma}}{\varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0 + \tau)},$$

respectively.

By inserting (3.151) and (3.152) into (3.149) and (3.150) respectively, to get

$$w_2(\omega) = \tilde{p}_2 + i\omega\tilde{\sigma}\left[\tilde{p}_2\varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0 + \tau) - \tilde{\sigma}\right]\frac{\tilde{G}(\tilde{p}_2)}{\tilde{F}'(\tilde{p}_2)} + O(\omega^2), \quad (3.153)$$

$$w_3(\omega) = \tilde{p}_3 + i\omega\tilde{\sigma}\left[\tilde{p}_3\varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0 + \tau) - \tilde{\sigma}\right]\frac{\tilde{G}(\tilde{p}_3)}{\tilde{F}'(\tilde{p}_3)} + O(\omega^2). \quad (3.154)$$

Summarising, branches $w_i(\omega)$, $i = 1, 2, 3$ are stable if the quantity $\tilde{p}_i\varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0 + \tau) - \tilde{\sigma}$ is negative. Conversely, these branches become unstable if this quantity is positive.

High frequency expansions

In order to get the roots of the secular equation (3.141) in the high frequency limits $\omega \rightarrow \infty$, we need to take $\omega^{-1} \rightarrow 0$. By putting $\omega = \frac{1}{\zeta}$, the secular equation (3.141) becomes

$$\tilde{F}(w) + \frac{\tilde{\sigma}}{(i\zeta + \alpha_0)} \left[w\varepsilon^{\frac{1}{2}} \left((\alpha_1 - \alpha_0) + \tau(1 - i\zeta^{-1}\alpha_0) \right) - \tilde{\sigma} \right] \tilde{G}(w) = 0, \quad (3.155)$$

Multiplying equation (3.155) by ζ we get

$$\zeta \tilde{F}(w) + \frac{\tilde{\sigma}}{(i\zeta + \alpha_0)} \left[w\varepsilon^{\frac{1}{2}} \left((\alpha_1 - \alpha_0 + \tau)\zeta - iw\varepsilon^{\frac{1}{2}}\alpha_0\tau \right) - \tilde{\sigma} \right] \tilde{G}(w) = 0. \quad (3.156)$$

Rearranging this equation as

$$\zeta \tilde{F}(w) + \frac{\tilde{\sigma}}{(i\zeta + \alpha_0)} \left[\zeta \left(w\varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0 + \tau) - \tilde{\sigma} \right) - i\alpha_0 w\varepsilon^{\frac{1}{2}}\tau \right] \tilde{G}(w) = 0. \quad (3.157)$$

Put $\zeta = 0$ to obtain

$$-i\tilde{\sigma}w\varepsilon^{\frac{1}{2}}\tau\tilde{G}(w) = 0. \quad (3.158)$$

So if $\tilde{\sigma} \neq 0$, the roots are $w_1 = \tilde{W}_1$, $w_2 = \tilde{W}_2$, $w_3 = 0$ and the last one $w_4 \rightarrow \infty$.

Now look at $\zeta \rightarrow 0$, so the roots of the secular equation (3.141) become

$$w_1 = \tilde{W}_1 + A\zeta + O(\zeta^2), \quad (3.158a)$$

$$w_2 = \tilde{W}_2 + B\zeta + O(\zeta^2). \quad (3.158b)$$

$$w_3 = C\zeta + O(\zeta^2). \quad (3.158c)$$

$$w_4 = D\zeta^{-1} + E + O(\zeta). \quad (3.158d)$$

In order to get these roots we need to find A, B, C, D, E .

Firstly by substituting (3.158a) into (3.157), we get

$$A = \frac{\tilde{W}_1(\tilde{W}_1 - \tilde{p}_1)(\tilde{W}_1 - \tilde{p}_2)(\tilde{W}_1 - \tilde{p}_3)}{i\tilde{\sigma}\tilde{W}_1\varepsilon^{\frac{1}{2}}\tau(\tilde{W}_1 - \tilde{W}_2)},$$

thus, the first root is

$$w_1 = \tilde{W}_1 - \left(\frac{i\zeta}{\tilde{\sigma}\tilde{W}_1\varepsilon^{\frac{1}{2}}\tau} \right) \frac{\tilde{F}(\tilde{W}_1)}{\tilde{G}'(\tilde{W}_1)} + O(\zeta^2).$$

Similarly, we can find that

$$B = \frac{\tilde{W}_2(\tilde{W}_2 - \tilde{p}_1)(\tilde{W}_2 - \tilde{p}_2)(\tilde{W}_2 - \tilde{p}_3)}{i\tilde{\sigma}\tilde{W}_2\varepsilon^{\frac{1}{2}}\tau(\tilde{W}_2 - \tilde{W}_1)},$$

so the second root becomes

$$w_2 = \tilde{W}_2 - \left(\frac{i\zeta}{\tilde{\sigma}\tilde{W}_2\varepsilon^{\frac{1}{2}}\tau} \right) \frac{\tilde{F}(\tilde{W}_2)}{\tilde{G}'(\tilde{W}_2)} + O(\zeta^2).$$

Baced on the inequalities (3.140a) we find that $A < 0$ and $B < 0$, and are not affected by the sign of \tilde{p}_1 , so $\text{Im } w_1(\zeta) > 0$ and $\text{Im } w_2(\zeta) > 0$, thus $w_1(\zeta)$ and $w_2(\zeta)$ are unstable in the high frequency.

Now inserting (3.158c) into (3.157) we get

$$C = 0, 0, 0, i\tilde{\sigma}\varepsilon^{1/2}\alpha_0^{-2}(\alpha_1 - \alpha_0 + \tau).$$

So, the third root may be written as

$$w_3 = i\zeta\tilde{\sigma}\varepsilon^{1/2}\alpha_0^{-2}(\alpha_1 - \alpha_0 + \tau) + O(\zeta^2).$$

The sign of imaginary part is positive, so w_3 is unstable in the high frequency.

Finally, substituting (3.158d) into (3.157) gives

$$D = 0, 0, 0, i\tilde{\sigma}\varepsilon^{1/2}\tau,$$

and

$$E = (\tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3) - (\tilde{W}_1 + \tilde{W}_2) - \tilde{\sigma}\alpha_0^{-1}\varepsilon^{1/2}(\alpha_1 - \alpha_0).$$

Thus, the fourth root becomes

$$w_4 = i\zeta^{-1}\tilde{\sigma}\varepsilon^{1/2}\tau + (\tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3) - (\tilde{W}_1 + \tilde{W}_2) - \tilde{\sigma}\alpha_0^{-1}\varepsilon^{1/2}(\alpha_1 - \alpha_0) + O(\zeta).$$

The stability condition is not satisfied here because $\text{Im } w_4(\zeta) > 0$, so w_4 is unstable in the high frequency.

It is clear that from Figures 3.9–3.11 branches can change their stability nature for intermediate frequencies. This is because we cannot rearrange (3.141) to have the real part in one side and the imaginary part on the other, as in equations (2.46) and (3.22).

The exceptional case $\tilde{p}_0 = 0$.

When $\tilde{p}_1 = 0$ the situation is as at the end of Section 3.3.1 and we have two branches beginning at the origin with arguments $-\pi/4$ and $3\pi/4$ in the complex w plane, see Figure 3.10.

Numerical results

In Figure 3.9 we have taken $\tilde{p}_1 > 0$. The branch $w_0(\omega)$ beginning at the origin is unstable in each part of the Figure. All the other branches begin to the right of this branch. If α_0 and α_1 are small enough then

$$\tilde{p}_i \varepsilon^{\frac{1}{2}} (\alpha_1 - \alpha_0 + \tau) - \tilde{\sigma} < 0, \quad \text{for } i = 1, 2, 3,$$

and so all the branches $w_i(\omega)$, $i = 1, 2, 3$, are stable in the low frequency. This can be seen in the first subfigures (a)–(c) of Figure 3.9 where α_0 and α_1 are small. As α_0 and α_1 increase, first $w_3(\omega)$ becomes unstable, see part (d), and as they increase further other branches become unstable.

In Figure 3.10 we illustrate the exceptional case $\tilde{p}_1 = 0$. Now two branches emanate from the origin, namely, $w_0(\omega)$ and $w_1(\omega)$, one stable and the other unstable, one with argument $-\pi/4$ and the other with argument $3\pi/4$. The same increasing instability with increasing α_0 and α_1 is observed except $w_0(\omega)$ retains its stability throughout the entire frequency range in each part of the Figure.

In Figure 3.11 we have taken $\tilde{p}_1 < 0$. The branch $w_1(\omega)$ beginning at $w = \tilde{p}_1$ is unstable in each part of the Figure. All the other branches begin to the right of this branch. The branch $w_0(\omega)$ begins at the origin and is stable in each part of the Figure. As in Figures 3.9 and 3.10, increasing α_0 and α_1 leads to increasing instability. As in the Figure 3.10 the branch $w_0(\omega)$ maintains its stability in the low and high frequencies in each part of the Figure.

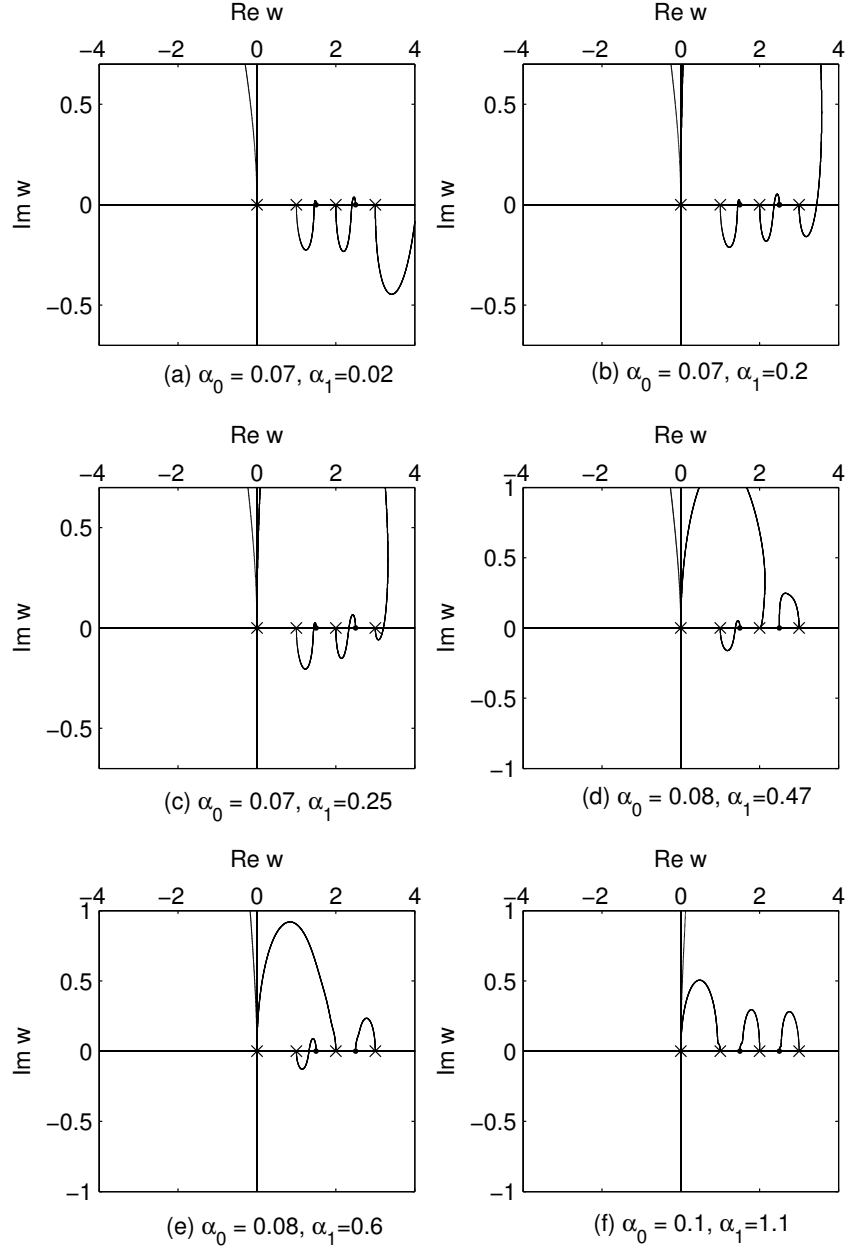


Figure 3.9: The longitudinal squared wave speeds of constrained anisotropic TRDTE+GTE (1). For each part, $\tilde{p}_1 = 1, \tilde{p}_2 = 2, \tilde{p}_3 = 3, \tilde{W}_1 = 1.5, \tilde{W}_2 = 2.5, \tilde{\sigma} = 1, \varepsilon = 1, \tau = 0.1$

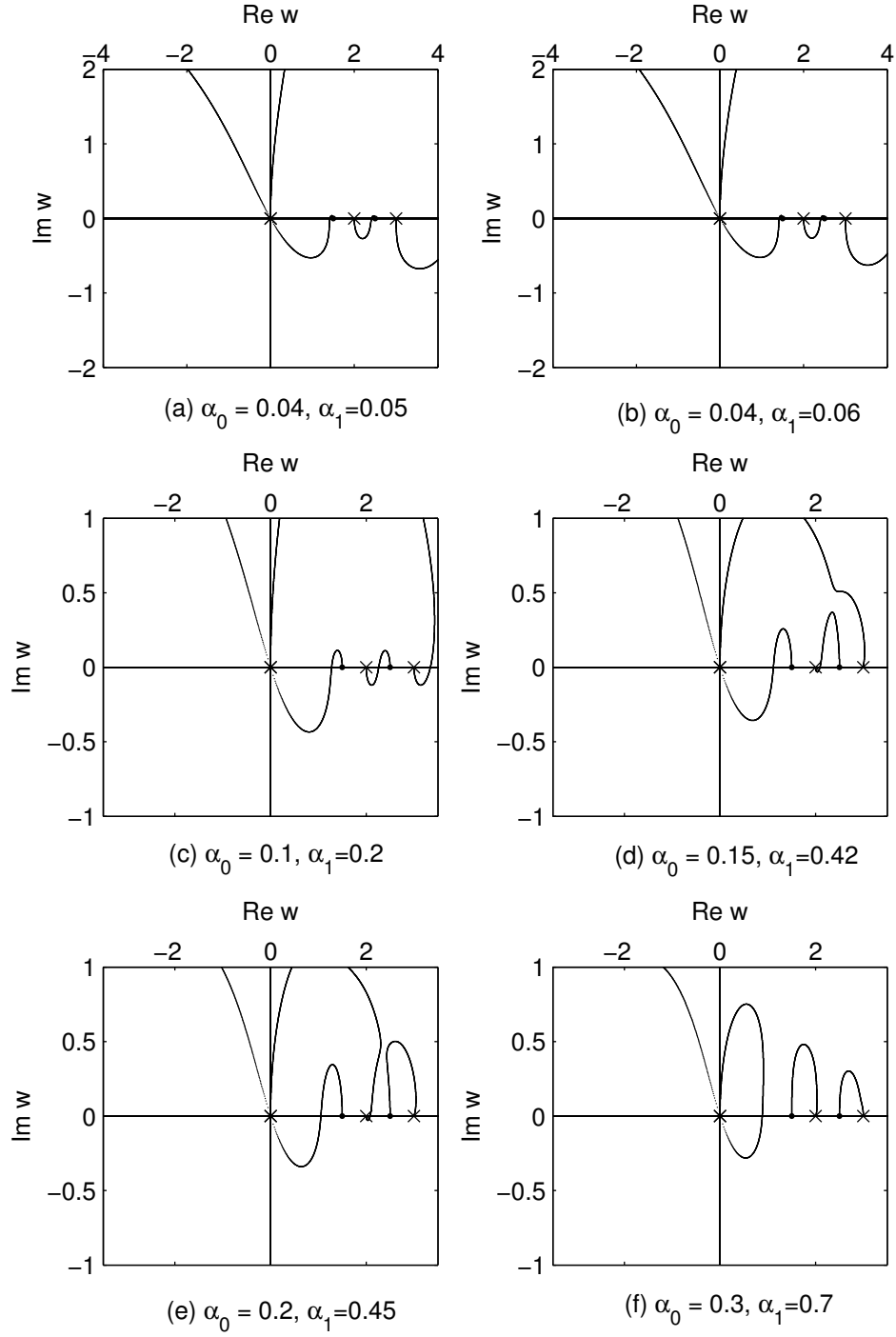


Figure 3.10: The longitudinal squared wave speeds of constrained anisotropic TRDTE+GTE (1). For each part, $\tilde{p}_1 = 0, \tilde{p}_2 = 2, \tilde{p}_3 = 3, \tilde{W}_1 = 1.5, \tilde{W}_2 = 2.5, \tilde{\sigma} = 1, \varepsilon = 1, \tau = 0.1$

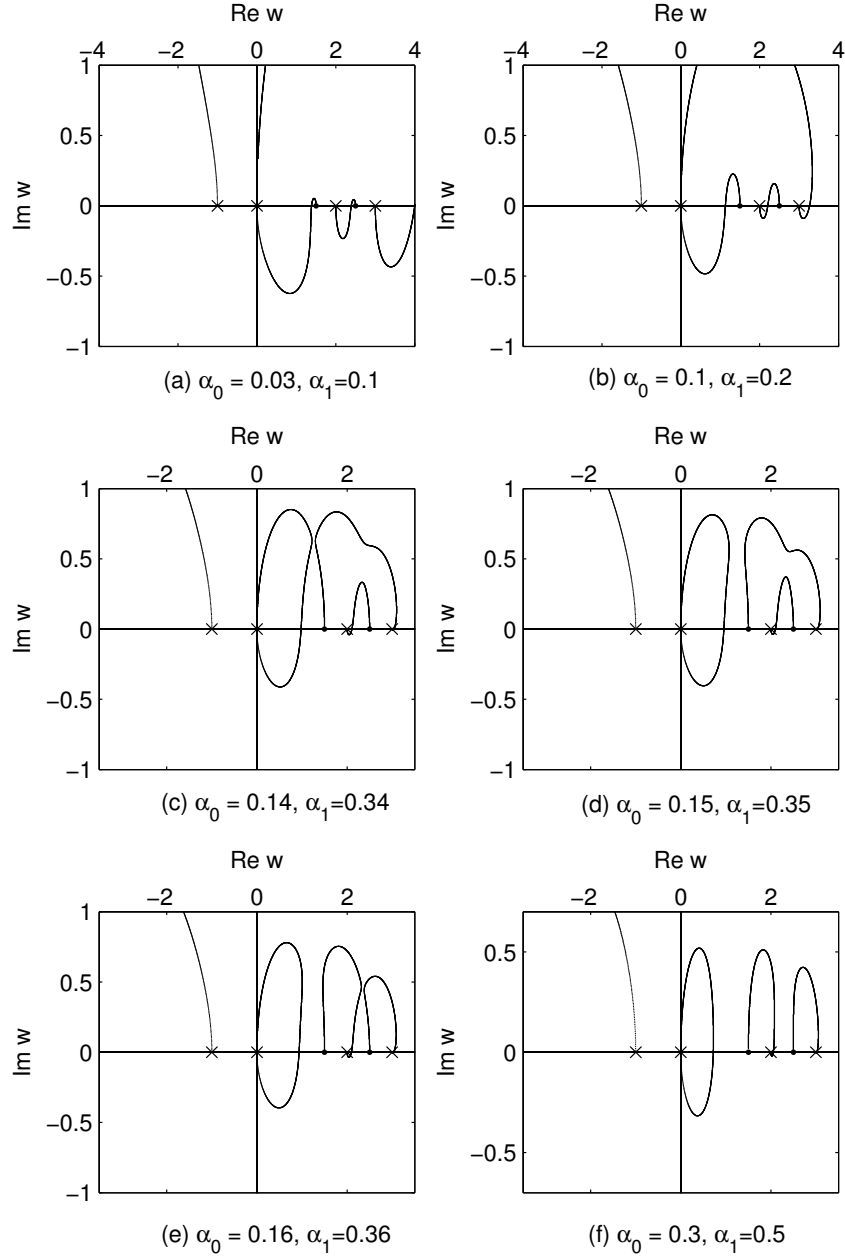


Figure 3.11: The longitudinal squared wave speeds of constrained anisotropic TRDTE+GTE (1). For each part, $\tilde{p}_1 = -1, \tilde{p}_2 = 2, \tilde{p}_3 = 3, \tilde{W}_1 = 1.5, \tilde{W}_2 = 2.5, \tilde{\sigma} = 1, \varepsilon = 1, \tau = 0.1$

3.4 Constrained isotropic TRDTE+GTE (1)

3.4.1 Usual form of deformation-temperature constraint

The field equations for TRDTE+GTE (1) of constrained anisotropic thermoelastic materials are (3.81). Then applying (2.191) to (3.81) we will get the field equations of constrained isotropic TRDTE+GTE (1) as

$$\left. \begin{aligned} (\tilde{\lambda} + \tilde{\mu})u_{j,ij} + \tilde{\mu}u_{i,jj} - \beta(\theta + \alpha_1\dot{\theta})_{,i} + \tilde{N}\tilde{\eta}_{,i} &= \rho\ddot{u}_i, \\ k\theta_{,ii} - T\beta(\dot{u}_{i,i} + \tau\ddot{u}_{i,i}) - \alpha T\dot{\tilde{\eta}} &= \rho c(\dot{\theta} + \alpha_0\ddot{\theta}), \\ \tilde{N}u_{i,i} - \alpha\theta &= 0. \end{aligned} \right\} \quad (3.159)$$

$$\tilde{\mathbf{N}} = \tilde{N}\mathbf{I}. \quad (3.159a)$$

The secular equation

Similarly to Section 2.4.1 we are seeking for solutions of (3.159) in the form of plane harmonic waves (2.86). Insert (2.86) into (3.159) with aid of (3.159a) to get the following system of algebraic equations

$$\left. \begin{aligned} [(\tilde{\mu} - \rho s^{-2})\delta_{ij} + (\tilde{\lambda} + \tilde{\mu})n_i n_j]U_j + i\beta(\omega s)^{-1}n_i(1 - i\omega\alpha_1)\Theta - i\tilde{N}n_i(\omega s)^{-1}\tilde{H} &= 0, \\ T\beta\omega s n_i(1 - i\omega\tau)U_i + (\omega s^2 k - i\rho c(1 - i\omega\alpha_0))\Theta - i\alpha T\tilde{H} &= 0, \\ \tilde{N}n_i i\omega s U_i - \alpha\Theta &= 0. \end{aligned} \right\} \quad (3.160)$$

It is required to eliminate Θ and H between (3.160). Firstly, from (3.160)₃ we can write it with subject Θ as follows

$$\Theta = \frac{i\omega s \tilde{N} n_i U_i}{\alpha}. \quad (3.161)$$

Substituting (3.161) into (3.160)₂, we get

$$\tilde{H} = -i\alpha^{-1}\beta\omega s n_i(1 - i\omega\tau)U_i + \left(\frac{\omega s \tilde{N} n_i}{\alpha^2 T}\right)(\omega s^2 k - i\rho c(1 - i\omega\alpha_0))U_i. \quad (3.162)$$

Inserting (3.161) and (3.162) into (3.160)₁, we get

$$\begin{aligned} &\left[(\tilde{\mu} - w)\delta_{ij} + (\tilde{\lambda} + \tilde{\mu})n_i n_j\right]U_j + i\beta(\omega s)^{-1}n_i(1 - i\omega\alpha_1)\left(\frac{i\omega s \tilde{N} n_j U_j}{\alpha}\right) - \\ &i\tilde{N}(\omega s)^{-1}n_i \left[-i\alpha^{-1}\beta\omega s n_j(1 - i\omega\tau)U_j + \frac{\omega s \tilde{N} n_j}{\alpha^2 T}(\omega s^2 k - i\rho c(1 - i\omega\alpha_0))U_j\right] = 0. \end{aligned}$$

After simplifying and rearranging the equation we obtain

$$\left\{ (\tilde{\mu} - w)\delta_{ij} + \left[\left(\tilde{\lambda} + \tilde{\mu} - \alpha^{-1}\beta\tilde{N}[(1 - i\omega\alpha_1) + (1 - i\omega\tau)] \right) - \left(\frac{\rho c \tilde{N}^2}{\alpha^2 T} \right) \left((1 - i\omega\alpha_0) + \frac{i\omega k}{wc} \right) \right] n_i n_j \right\} U_j = 0, \quad (3.163)$$

which gives in direct notation the secular equation

$$\det \left\{ (\tilde{\mu} - w)\mathbf{1} + \left[\left(\tilde{\lambda} + \tilde{\mu} - \alpha^{-1}\beta\tilde{N}[(1 - i\omega\alpha_1) + (1 - i\omega\tau)] \right) - \left(\frac{\rho c \tilde{N}^2}{\alpha^2 T} \right) \left((1 - i\omega\alpha_0) + \frac{i\omega k}{wc} \right) \right] \mathbf{n} \otimes \mathbf{n} \right\} = 0. \quad (3.164)$$

Non-dimensionalize this equation by applying the dimensionless quantities (2.57) and further dimensionless quantities

$$\alpha' = \alpha T, \quad c' = \rho c T / \gamma, \quad \omega^* = \gamma c / k,$$

to get

$$\det \left\{ (\tilde{\mu}' - w')\mathbf{1} + \left[\left(\tilde{\lambda}' + \tilde{\mu}' - \frac{(\varepsilon c')^{1/2} \tilde{N}}{\alpha'} [(1 - i\omega' \alpha'_1) + (1 - i\omega\tau)] \right) - \left(\frac{c' \tilde{N}^2}{\alpha'^2} \right) \left((1 - i\omega' \alpha'_0) + \frac{i\omega'}{w'} \right) \right] \mathbf{n} \otimes \mathbf{n} \right\} = 0. \quad (3.165)$$

Now by using the standard identity (2.60), and dropping dashes for convenience, we get the secular equation as follows

$$(w - \tilde{\mu})^2 \left[w^2 - w \left(1 - (\varepsilon c)^{1/2} \tilde{N} \alpha^{-1} [(1 - i\omega\alpha_1) + (1 - i\omega\tau)] - c \tilde{N}^2 \alpha^{-2} (1 - i\omega\alpha_0) \right) + i\omega c \tilde{N}^2 \alpha^{-2} \right] = 0. \quad (3.166)$$

This is the secular equation for isotropic TRDTE+GTE (1) which is constrained by the usual deformation temperature constraint and has not previously appeared in the literature.

The repeated root $w = \tilde{\mu}$ represents two purely elastic transverse waves, and longitudinal waves are represented by roots of the following quadratic equation

$$\alpha^2 w^2 - w \left(\alpha^2 - (\varepsilon c)^{1/2} \alpha \tilde{N} [(1 - i\omega\alpha_1) + (1 - i\omega\tau)] - c \tilde{N}^2 (1 - i\omega\alpha_0) \right) + i\omega c \tilde{N}^2 = 0. \quad (3.167)$$

Equation (3.167) may be written as

$$w^2 - w \left(1 - \varepsilon^{1/2} \tilde{\sigma} [(1 - i\omega\alpha_1) + (1 - i\omega\tau)] - \tilde{\sigma}^2 (1 - i\omega\alpha_0) \right) + i\omega\tilde{\sigma}^2 = 0. \quad (3.168)$$

The secular equation (3.168) may be written as

$$w^2 - w \left\{ 1 - \varepsilon^{1/2} \tilde{\sigma} [2 - i\omega(\alpha_1 + \tau)] - \tilde{\sigma}^2 (1 - i\omega\alpha_0) \right\} + i\omega\tilde{\sigma}^2 = 0. \quad (3.168a)$$

where $\tilde{\sigma}$ is defined earlier in (2.202). In this equation there are two parameters, ε and σ , which affect the behaviour of the roots of this equation. Returning to the special cases that are discussed in Section 2.4.1, namely, $(\tilde{N} = 0, \alpha \neq 0)$ and $(\tilde{N} \neq 0, \alpha = 0)$, we find that the solutions of (3.168a) reduces to equations (2.204) and (2.205), respectively. But in general, with neither \tilde{N} nor α is equal to zero, it is convenient to go back to equation (3.168). The roots of (3.168) are

$$w_{1,2} = \bar{A} \pm \left[\bar{A}^2 - i\omega\tilde{\sigma}^2 \right]^{\frac{1}{2}}, \quad (3.169)$$

where

$$\bar{A} = \frac{1}{2} \left[1 - \varepsilon^{1/2} \tilde{\sigma} [(1 - i\omega\alpha_1) + (1 - i\omega\tau)] - \tilde{\sigma}^2 (1 - i\omega\alpha_0) \right]. \quad (3.170)$$

This equation may be written as

$$\bar{A} = \frac{1}{2} \left[1 - 2\varepsilon^{1/2} \tilde{\sigma} - \tilde{\sigma}^2 + i\omega\tilde{\sigma} (\varepsilon^{1/2}(\alpha_1 + \tau) + \alpha_0\tilde{\sigma}^2) \right].$$

For fixed $\varepsilon \geq 0$, as $\tilde{\sigma}$ increases from 0 to ∞ , $\text{Re } \bar{A}$ decreases from $\frac{1}{2}$ to $-\infty$. $\text{Re } \bar{A}$ becomes 0 at $\omega = 0$ for a critical value of $\tilde{\sigma}$ given by

$$\tilde{\sigma}_c = (1 + \varepsilon)^{1/2} - \varepsilon^{1/2}. \quad (3.171)$$

In the special case where $\tilde{\sigma} = \tilde{\sigma}_c$, so $\text{Re } \bar{A} = 0$ in (3.169), we get

$$w = \pm e^{i\frac{\pi}{4}} \omega^{1/2} \tilde{\sigma}_c + O(\omega), \quad (3.171a)$$

and this is similar to its counterpart (2.208) in TRDTE.

Low frequency expansions

The roots of the secular equation (3.168a) in the low frequency limit when $\omega = 0$ are

$$w_1 = 1 - 2\varepsilon^{1/2} - \tilde{\sigma}^2, \quad w_2 = 0.$$

But as $\omega \rightarrow 0$ the roots of the secular equation take the form

$$w_1 = 1 - 2\varepsilon^{1/2} - \tilde{\sigma}^2 + A(i\omega) + O(\omega^2), \quad w_2 = B(i\omega) + O(\omega^2). \quad (3.171b)$$

Inserting (3.171b) into (3.168a) we get

$$A = \tilde{\sigma} [\alpha_0 \tilde{\sigma} + \varepsilon^{1/2}(\alpha_1 + \tau)] - \frac{\tilde{\sigma}^2}{1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2} \quad \text{and} \quad B = \frac{\tilde{\sigma}^2}{1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2}.$$

Thus the roots become

$$w_1 = 1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2 + i\omega\tilde{\sigma} \left\{ \alpha_0 \tilde{\sigma} + \varepsilon^{1/2}(\alpha_1 + \tau) - \frac{\tilde{\sigma}}{1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2} \right\} + O(\omega^2),$$

and

$$w_2 = \frac{i\omega\tilde{\sigma}^2}{1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2} + O(\omega^2).$$

If $\tilde{\sigma} < \tilde{\sigma}_c$, we cannot tell about the sign of $\text{Im } w_1(\omega)$ because it depends on the relative values of the quantities occurring but it is clear that $\text{Im } w_2(\omega) > 0$ so w_2 is unstable. If $\tilde{\sigma} > \tilde{\sigma}_c$, $\text{Im } w_1(\omega) > 0$ so w_1 is unstable and $\text{Im } w_2(\omega) < 0$ so w_2 is stable. If $\tilde{\sigma} = \tilde{\sigma}_c$ the analysis is not valid and will return to the roots of the secular equation in the special case (3.171a).

High frequency expansions

In the high frequency limits as $\omega \rightarrow \infty$, $(i\omega)^{-1} \rightarrow 0$. The secular equation (3.168a) after dividing by $i\omega$ becomes

$$w^2(i\omega)^{-1} - w \left\{ (i\omega)^{-1} - 2\varepsilon^{1/2}\tilde{\sigma}(i\omega)^{-1} + \varepsilon^{1/2}\tilde{\sigma}(\alpha_1 + \tau) - \tilde{\sigma}^2(i\omega)^{-1} + \tilde{\sigma}^2\alpha_0 \right\} + \tilde{\sigma}^2 = 0. \quad (3.171c)$$

The roots of the secular equation (3.168a) in the high frequency may obtained by putting $(i\omega)^{-1} = 0$ and we get

$$w_1 = \frac{\tilde{\sigma}}{\varepsilon^{1/2}(\alpha_1 + \tau) + \alpha_0\tilde{\sigma}}, \quad \text{and} \quad w_2 \rightarrow \infty.$$

Now the roots of the secular equation (3.171c) as $(i\omega)^{-1} \rightarrow 0$ are given by

$$w_1 = \frac{\tilde{\sigma}}{\varepsilon^{1/2}(\alpha_1 + \tau) + \alpha_0\tilde{\sigma}} + A(i\omega)^{-1} + O(\omega^{-2}), \quad \text{and} \quad w_2 = B(i\omega) + C + O(\omega^{-1}). \quad (3.171d)$$

By substituting (3.171d) into (3.171c) we obtain

$$A = \frac{1}{(\varepsilon^{1/2}(\alpha_1 + \tau) + \alpha_0\tilde{\sigma})^2} \left[\frac{\tilde{\sigma}}{\varepsilon^{1/2}(\alpha_1 + \tau) + \alpha_0\tilde{\sigma}} - (1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2) \right],$$

$$B = \tilde{\sigma}(\varepsilon^{1/2}(\alpha_1 + \tau) + \tilde{\sigma}\alpha_0), \quad \text{and} \quad C = 1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2 - \frac{\tilde{\sigma}}{(\varepsilon^{1/2}(\alpha_1 + \tau) + \tilde{\sigma}\alpha_0)}.$$

Thus the roots become

$$w_1 = \frac{\tilde{\sigma}}{\varepsilon^{1/2}(\alpha_1 + \tau) + \alpha_0\tilde{\sigma}} - \frac{i\omega^{-1}}{(\varepsilon^{1/2}(\alpha_1 + \tau) + \alpha_0\tilde{\sigma})^2} \left[\frac{\tilde{\sigma}}{\varepsilon^{1/2}(\alpha_1 + \tau) + \alpha_0\tilde{\sigma}} - (1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2) \right] + O(\omega^{-2}). \quad (3.171e)$$

If $\tilde{\sigma} < \tilde{\sigma}_c$, we cannot tell about the sign of $\text{Im } w_1(\omega)$ because it depends on the relative values of the quantities occurring, but if $\tilde{\sigma} > \tilde{\sigma}_c$, $\text{Im } w_1(\omega) < 0$ so w_1 is stable. If $\tilde{\sigma} = \tilde{\sigma}_c$ we find that $\text{Im } w_1(\omega) < 0$, thus w_1 is stable.

$$w_2 = i\omega\tilde{\sigma}(\varepsilon^{1/2}(\alpha_1 + \tau) + \tilde{\sigma}\alpha_0) + 1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2 - \frac{\tilde{\sigma}}{(\varepsilon^{1/2}(\alpha_1 + \tau) + \tilde{\sigma}\alpha_0)} + O(\omega^{-1}) \quad (3.171f)$$

It is clear that $\text{Im } w_2(\omega) > 0$, so w_2 is unstable in the high frequency.

Now we consider the two previous special cases.

Case 1: The isothermal constraint viewed as the limit $\tilde{\sigma} \rightarrow 0$

Expanding (3.169) we obtain

$$w_{1,2} = \frac{1}{2} \left(1 - \varepsilon^{1/2}\tilde{\sigma}((1 - i\omega\alpha_1) + (1 - i\omega\tau)) - \tilde{\sigma}^2(1 - i\omega\alpha_0) \right) \pm \frac{1}{2} \left\{ \left(1 - \varepsilon^{1/2}\tilde{\sigma}((1 - i\omega\alpha_1) + (1 - i\omega\tau)) - \tilde{\sigma}^2(1 - i\omega\alpha_0) \right)^2 - 4i\omega\tilde{\sigma}^2 \right\}^{1/2}.$$

After expanding and using the binomial expansion we get

$$w_1 = 1 - \varepsilon^{1/2}\tilde{\sigma}((1 - i\omega\alpha_1) + (1 - i\omega\tau)) - \tilde{\sigma}^2(1 - i\omega\alpha_0) - i\omega\tilde{\sigma}^2 + O(\tilde{\sigma}^3). \quad (3.172)$$

Rearrange this equation as

$$w_1 = 1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2 + i\omega\tilde{\sigma}[\varepsilon^{1/2}(\alpha_1 + \tau) + \alpha_0\tilde{\sigma} - \tilde{\sigma}] + O(\tilde{\sigma}^3),$$

$$w_2 = i\omega\tilde{\sigma}^2 + O(\tilde{\sigma}^3). \quad (3.173)$$

Equation (3.173) is similar to its counterpart (2.210) in Section 2.4.1.

Case 2: The purely mechanical constraint viewed as the limit $\tilde{\sigma} \rightarrow \infty$

When $\tilde{\sigma} \rightarrow \infty$ means $\frac{1}{\tilde{\sigma}}$ is small, from (3.169), after expanding and factorising by

$\tilde{\sigma}^4(1 - i\omega\alpha_0)$, then using the binomial expansion, we obtain

$$w_1 = \frac{-i\omega}{(1 - i\omega\alpha_0)} \left[1 - \varepsilon^{1/2} \tilde{\sigma}^{-1} \frac{((1 - i\omega\alpha_1) + (1 - i\omega\tau))}{(1 - i\omega\alpha_0)} \right] + O(\tilde{\sigma}^{-2}). \quad (3.174)$$

$$w_2 = 1 - \varepsilon^{1/2} \tilde{\sigma}((1 - i\omega\alpha_1) + (1 - i\omega\tau)) - \tilde{\sigma}^2(1 - i\omega\alpha_0) + \frac{i\omega}{(1 - i\omega\alpha_0)} \left[1 - \varepsilon^{1/2} \tilde{\sigma}^{-1} \frac{((1 - i\omega\alpha_1) + (1 - i\omega\tau))}{(1 - i\omega\alpha_0)} \right] + O(\tilde{\sigma}^{-2}). \quad (3.175)$$

Putting $\alpha_0 = \alpha_1 = 0$ and $\tau = 0$ in equations (3.119)–(3.175) returns us to classical thermoelasticity.

Numerical results

In each of Figures 3.12 and 3.13 we use $\varepsilon = 1$, but in Figure 3.12 we choose $\alpha_0 = 0.01$ and $\alpha_1 = 0.02$, and in Figure 3.13 $\alpha_0 = 0.1$ and $\alpha_1 = 0.2$, and w is plotted for a range of values of $\tilde{\sigma}$. We have two longitudinal waves one is stable and finite and the other is unstable and tends to infinity. The low frequency limits are marked by a \times and the high frequency limits are marked by a \circ . The unstable branch starting from the origin and the stable branch starting from $1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2$, when $\tilde{\sigma} < \tilde{\sigma}_c$ see subfigures (a)–(c). But as $\tilde{\sigma} > \tilde{\sigma}_c$ the situation is reversed, see subfigures (e) and (f). In the special case when $\tilde{\sigma} = \tilde{\sigma}_c$ the branches become a connected line passing through the origin at angle $-\pi/4$ to the real axis, see the subfigure (d). It is clear that increasing of α_0 and α_1 does not change branches stability nature.

Varying the parameters τ and ε while changing the magnitude of ω does not have any substantive influence on the stability.

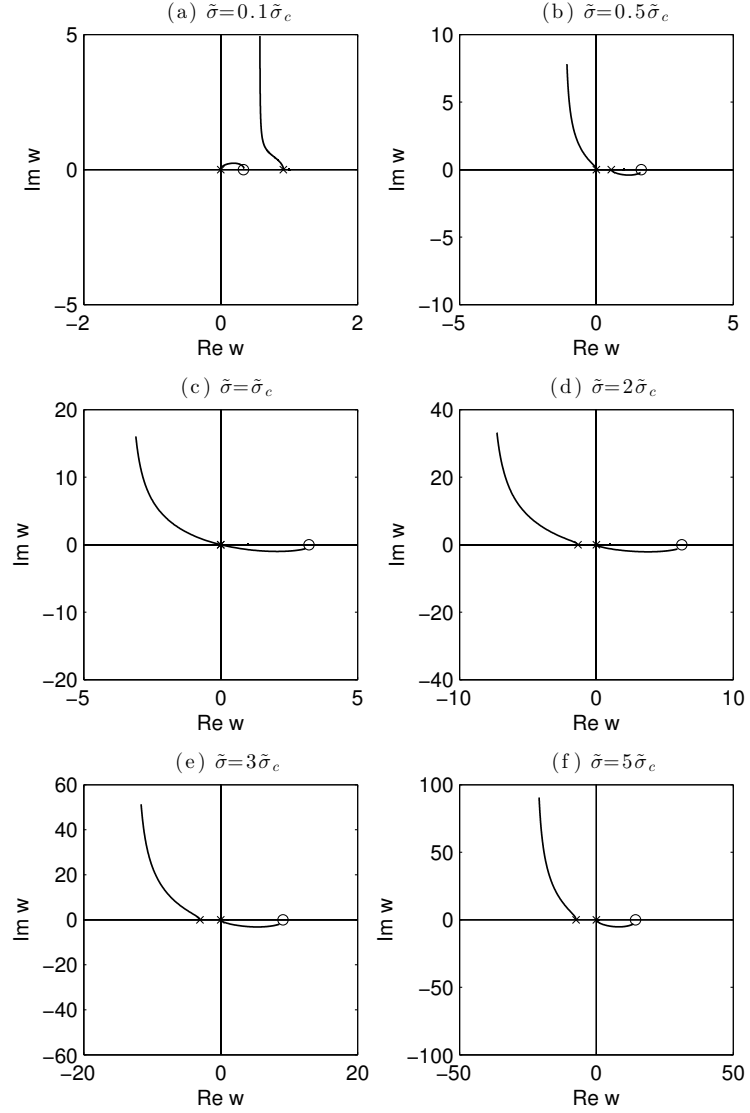


Figure 3.12: The longitudinal squared wave speeds of isotropic thermelastic material for TRDTE+GTE(1) theory with incompressibility at uniform temperature. For each part ($\varepsilon = 1, \alpha_0 = 0.01, \alpha_1 = 0.02, \tau = 0.1$), (a) $\tilde{\sigma} = 0.1\tilde{\sigma}_c$, (b) $\tilde{\sigma} = 0.5\tilde{\sigma}_c$, (c) $\tilde{\sigma} = \tilde{\sigma}_c$, (d) $\tilde{\sigma} = 2\tilde{\sigma}_c$, (e) $\tilde{\sigma} = 3\tilde{\sigma}_c$, (f) $\tilde{\sigma} = 5\tilde{\sigma}_c$.

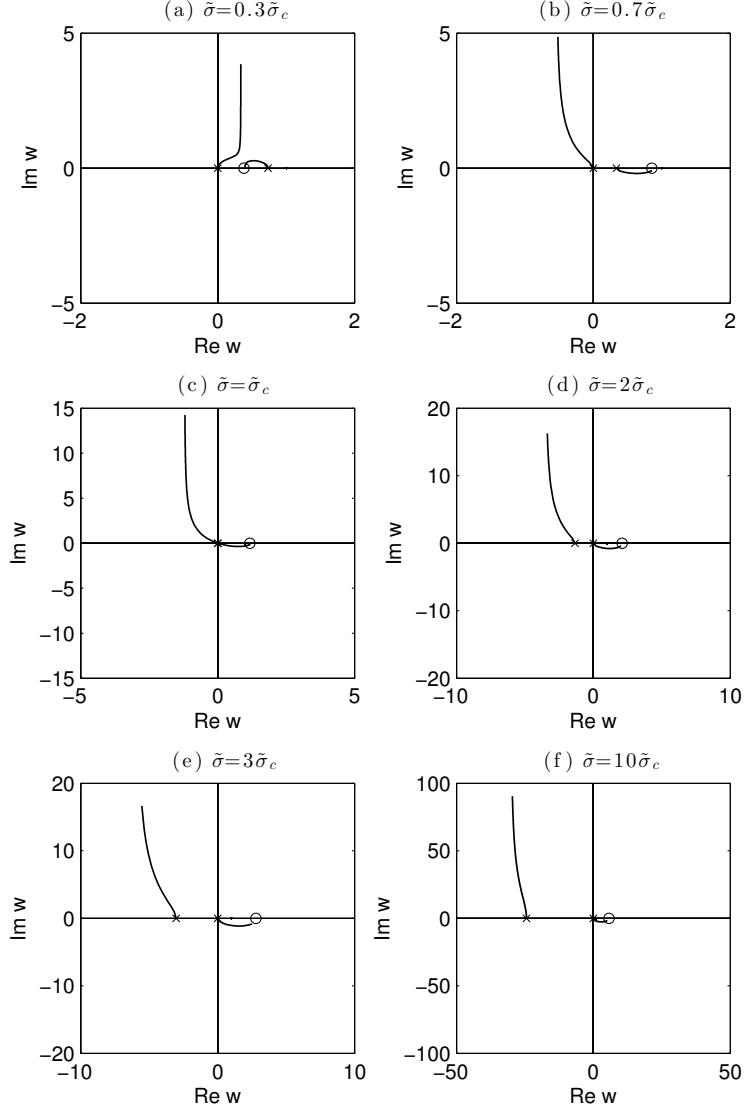


Figure 3.13: The longitudinal squared wave speeds of isotropic thermelastic material for TRDTE+GTE theory with incompressibility at uniform temperature. For each part ($\varepsilon = 1, \alpha_0 = 0.1, \alpha_1 = 0.2, \tau = 0.1$), (a) $\tilde{\sigma} = 0.3\tilde{\sigma}_c$, (b) $\tilde{\sigma} = 0.7\tilde{\sigma}_c$, (c) $\tilde{\sigma} = \tilde{\sigma}_c$, (d) $\tilde{\sigma} = 2\tilde{\sigma}_c$, (e) $\tilde{\sigma} = 3\tilde{\sigma}_c$, (f) $\tilde{\sigma} = 10\tilde{\sigma}_c$.

3.4.2 Alternative form of deformation-temperature constraint

In this section we will use equations (3.159)_{1,2} and the alternative form of deformation temperature constraint (2.144), to get the field equations of TRDTE + GTE (1) for an isotropic material incompressible at fixed temperature

$$\left. \begin{aligned} (\tilde{\lambda} + \tilde{\mu})u_{j,ij} + \tilde{\mu}u_{i,jj} - \beta\left(1 + \alpha_1\frac{\partial}{\partial t}\right)\theta_{,i} + \tilde{N}\tilde{\eta}_{,i} &= \rho\ddot{u}_i, \\ k\theta_{,ii} - T\beta\left(1 + \tau\frac{\partial}{\partial t}\right)\dot{u}_{i,i} - \alpha T\dot{\tilde{\eta}} &= \rho c\left(1 + \alpha_0\frac{\partial}{\partial t}\right)\dot{\theta}, \\ \tilde{N}u_{i,i} - \alpha(\theta + \alpha_0\dot{\theta}) &= 0. \end{aligned} \right\} \quad (3.176)$$

The secular equation

Now we are looking for solutions in the form of plane harmonic waves (2.86), by inserting (2.86) into (3.176) with aid (3.159a) we get the system of algebraic equations

$$\left. \begin{aligned} [(\tilde{\mu} - \rho s^{-2})\delta_{ij} + (\tilde{\lambda} + \tilde{\mu})n_i n_j]U_i + i\beta(\omega s)^{-1}n_i(1 - i\omega\alpha_1)\Theta - i\tilde{N}n_j(\omega s)^{-1}\tilde{H} &= 0, \\ T\beta\omega s n_i(1 - i\omega\tau)U_i + [\omega s^2 k - i\rho c(1 - i\omega\alpha_0)]\Theta - i\alpha T\tilde{H} &= 0, \\ \tilde{N}n_i i\omega s U_i - \alpha(1 - i\omega\alpha_0)\Theta &= 0. \end{aligned} \right\} \quad (3.177)$$

Eliminate Θ and \tilde{H} between (3.177), similarly to (3.160). From (3.177)₃ we can write Θ as follows

$$\Theta = \frac{i\omega s \tilde{N} n_i U_i}{\alpha(1 - i\omega\alpha_0)} \quad (3.178)$$

Substituting (3.178) into (3.177)₂, we get

$$\tilde{H} = -i\alpha^{-1}\beta(1 - i\omega\tau)\omega s n_i U_i + [\omega s^2 k - i\rho c(1 - i\omega\alpha_0)]\left(\frac{\omega s \tilde{N} n_i U_i}{\alpha^2 T(1 - i\omega\alpha_0)}\right). \quad (3.179)$$

Inserting Θ and \tilde{H} into (3.177)₁ we get after simplifying

$$\left\{ (\tilde{\mu} - w)\delta_{ij} + \left[\tilde{\lambda} + \tilde{\mu} - \alpha^{-1}\beta\tilde{N}\left((1 - i\omega\tau) + \frac{1 - i\omega\alpha_1}{1 - i\omega\alpha_0}\right) \right] - \frac{\rho c \tilde{N}^2}{\alpha^2 T(1 - i\omega\alpha_0)} \left[(1 - i\omega\alpha_0) + \frac{i\omega k}{wc} \right] n_i n_j \right\} U_i = 0. \quad (3.180)$$

This gives in direct notation the secular equation

$$\det \left\{ (\tilde{\mu} - w)\mathbf{1} + \left[\tilde{\lambda} + \tilde{\mu} - \alpha^{-1}\beta\tilde{N}\left((1 - i\omega\tau) + \frac{1 - i\omega\alpha_1}{1 - i\omega\alpha_0}\right) - \frac{\rho c \tilde{N}^2}{\alpha^2 T(1 - i\omega\alpha_0)} \left[(1 - i\omega\alpha_0) + \frac{i\omega k}{wc} \right] \mathbf{n} \otimes \mathbf{n} \right] \right\} = 0. \quad (3.181)$$

Non-dimensionalize this equation by applying the dimensionless quantities (2.57) to get

$$\det \left\{ (\tilde{\mu}' - w') \mathbf{1} + \left[\tilde{\lambda}' + \tilde{\mu}' - \frac{(\varepsilon c')^{1/2}}{\alpha'} \tilde{N} \left((1 - i\omega' \tau') + \frac{1 - i\omega' \alpha'_1}{1 - i\omega' \alpha'_0} \right) - \frac{c' \tilde{N}^2}{\alpha'^2 (1 - i\omega' \alpha'_0)} \left[(1 - i\omega' \alpha'_0) + \frac{i\omega'}{w'} \right] \mathbf{n}' \otimes \mathbf{n}' \right\} = 0. \quad (3.182)$$

Now by using the standard identity (2.60), dropping the dashes for convenience, we get the secular equation as follows

$$(w - \tilde{\mu})^2 \left\{ \alpha^2 w^2 - w \left[\alpha^2 - (\varepsilon c)^{1/2} \alpha \tilde{N} \left[(1 - i\omega \tau) + \frac{1 - i\omega \alpha_1}{1 - i\omega \alpha_0} \right] + c \tilde{N}^2 \right] - \frac{i\omega c \tilde{N}^2}{1 - i\omega \alpha_0} \right\} = 0. \quad (3.183)$$

This is the secular equation for isotropic TRDTE+GTE (1) which is constrained by the alternative deformation temperature constraint and has not previously appeared in the literature.

The repeated root $w = \tilde{\mu}$ represents two purely elastic transverse waves. The longitudinal waves are roots of the following quadratic equation

$$\alpha^2 w^2 - w \left\{ \alpha^2 - (\varepsilon c)^{1/2} \alpha \tilde{N} \left[(1 - i\omega \tau) + \frac{1 - i\omega \alpha_1}{1 - i\omega \alpha_0} \right] - c \tilde{N}^2 \right\} + \frac{i\omega c \tilde{N}^2}{1 - i\omega \alpha_0} = 0. \quad (3.184)$$

By dividing by α^2 we get

$$w^2 - w \left\{ 1 - \frac{(\varepsilon c)^{1/2}}{\alpha} \tilde{N} \left[(1 - i\omega \tau) + \frac{1 - i\omega \alpha_1}{1 - i\omega \alpha_0} \right] - \frac{c \tilde{N}^2}{\alpha^2} \right\} + \frac{i\omega c \tilde{N}^2}{\alpha^2 (1 - i\omega \alpha_0)} = 0. \quad (3.185)$$

We can rewrite equation (3.185) as

$$w^2 - w \left\{ 1 - \varepsilon^{1/2} \tilde{\sigma} \left[(1 - i\omega \tau) + \frac{1 - i\omega \alpha_1}{1 - i\omega \alpha_0} \right] - \tilde{\sigma}^2 \right\} + \frac{i\omega \tilde{\sigma}^2}{1 - i\omega \alpha_0} = 0, \quad (3.186)$$

where $\tilde{\sigma}$ is defined earlier in (2.202). Equation (3.186) is the final form which represents the squared wave speeds of purely longitudinal waves propagating in an isotropic thermoelastic material that is incompressible at uniform temperature. Again going back to the special cases that are discussed in Section 2.4.1, ($\tilde{N} = 0, \alpha \neq 0$) and ($\tilde{N} \neq 0, \alpha = 0$), we will get the same results and analysis. In examining the more

general case in which neither \tilde{N} nor α is equal to zero, it is convenient to go back to equation (3.186). The roots of (3.186) are

$$w_{1,2} = \bar{A} \pm \left[\bar{A}^2 - \frac{i\omega\tilde{\sigma}^2}{1 - i\omega\alpha_0} \right]^{\frac{1}{2}}, \quad (3.187)$$

where

$$\bar{A} = \frac{1}{2} \left\{ 1 - \varepsilon^{1/2} \tilde{\sigma} \left[(1 - i\omega\tau) + \frac{1 - i\omega\alpha_1}{1 - i\omega\alpha_0} \right] - \tilde{\sigma}^2 \right\}. \quad (3.188)$$

Equation (3.188) may be written as

$$\bar{A} = \frac{1}{2} \left\{ 1 - \varepsilon^{1/2} \tilde{\sigma} - \tilde{\sigma}^2 - \frac{\varepsilon^{1/2} \tilde{\sigma}}{1 + \omega^2 \alpha_0^2} (1 + \alpha_1 \alpha_0 \omega^2) + i\omega \varepsilon^{1/2} \tilde{\sigma} \left[\tau + \frac{\alpha_1 + \alpha_0}{1 + \omega^2 \alpha_0^2} \right] \right\}.$$

For fixed $\varepsilon \geq 0$, as $\tilde{\sigma}$ increases from 0 to ∞ , $\text{Re } \bar{A}$ at $\omega = 0$ decreases from $\frac{1}{2}$ to $-\infty$. $\text{Re } \bar{A}$ becomes 0 at $\omega = 0$ for a critical value of $\tilde{\sigma}$ given by

$$\tilde{\sigma}_c = (1 + \varepsilon)^{1/2} - \varepsilon^{1/2}. \quad (3.189)$$

In the special case where $\tilde{\sigma} = \tilde{\sigma}_c$, so $\text{Re } \bar{A} = 0$ at $\omega = 0$ in (3.187), we get

$$\begin{aligned} w &= \pm \left(-i\omega \tilde{\sigma}_c^2 \right)^{\frac{1}{2}} + O(\omega), \\ &= \pm \left(e^{-i\frac{\pi}{2}} \tilde{\sigma}_c^2 \omega \right)^{\frac{1}{2}} + O(\omega), \end{aligned}$$

So,

$$w = \pm e^{-i\frac{\pi}{4}} \omega^{\frac{1}{2}} \tilde{\sigma}_c + O(\omega). \quad (3.190)$$

Low frequency expansions

The roots of the secular equation (3.186) in the low frequency limits at $\omega = 0$ are

$$w_1 = 1 - 2\varepsilon^{1/2} \tilde{\sigma} - \tilde{\sigma}^2, \quad \text{and} \quad w_2 = 0.$$

While the roots at $\omega \rightarrow 0$ may be written as

$$w_1 = 1 - 2\varepsilon^{1/2} \tilde{\sigma} - \tilde{\sigma}^2 + A(i\omega) + O(\omega^2), \quad \text{and} \quad w_2 = B(i\omega) + O(\omega^2). \quad (3.190a)$$

Substituting (3.190a) into (3.186) we get

$$A = \varepsilon^{1/2} \tilde{\sigma} (\alpha_1 - \alpha_0 + \tau) - \frac{\tilde{\sigma}^2}{1 - 2\varepsilon^{1/2} \tilde{\sigma} - \tilde{\sigma}^2}, \quad \text{and} \quad B = \frac{\tilde{\sigma}^2}{1 - 2\varepsilon^{1/2} \tilde{\sigma} - \tilde{\sigma}^2}.$$

Thus, the roots become

$$w_1 = 1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2 + i\omega\tilde{\sigma}\left\{\varepsilon^{1/2}\tilde{\sigma}(\alpha_1 - \alpha_0 + \tau) - \frac{\tilde{\sigma}}{1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2}\right\} + O(\omega^2),$$

and

$$w_2 = \frac{i\omega\tilde{\sigma}^2}{1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2} + O(\omega^2).$$

If $\tilde{\sigma} < \tilde{\sigma}_c$ we cannot tell about the sign of $\text{Im } w_1(\omega)$ because it depends on the relative values of the quantities occurring, but it is clear that $\text{Im } w_2(\omega) > 0$, so w_2 is unstable. If $\tilde{\sigma} > \tilde{\sigma}_c$ the sign of $\text{Im } w_1(\omega)$ is positive so w_1 is unstable and the sign of $\text{Im } w_2(\omega)$ is negative, so w_2 is stable. If $\tilde{\sigma} = \tilde{\sigma}_c$ the analysis is not valid and we can return in this case to (3.190).

High frequency expansions

In the high frequency as $\omega \rightarrow \infty$, $(\omega)^{-1} \rightarrow 0$. The secular equation (3.186) may be written as

$$w^2(1 - i\omega\alpha_0) - w\left\{(1 - i\omega\alpha_0) - \varepsilon^{1/2}\tilde{\sigma}[(1 - i\omega\tau)(1 - i\omega\alpha_0) + (1 - i\omega\alpha_1)] - \tilde{\sigma}^2(1 - i\omega\alpha_0)\right\} + i\omega\tilde{\sigma}^2 = 0. \quad (3.190b)$$

After dividing by $(i\omega)^2$ we get

$$w^2[(i\omega)^{-2} - (i\omega)^{-1}\alpha_0] - w\left\{[(i\omega)^{-2} - (i\omega)^{-1}\alpha_0] - \varepsilon^{1/2}\tilde{\sigma}[2(i\omega)^{-2} - (i\omega)^{-1}(\alpha_1 + \alpha_0 + \tau) + \tau\alpha_0] - \tilde{\sigma}^2[(i\omega)^{-2} - (i\omega)^{-1}\alpha_0]\right\} + (i\omega)^{-1}\tilde{\sigma}^2 = 0. \quad (3.190c)$$

Putting $(i\omega)^{-1} = 0$ into (3.190c) to get the roots of the secular equation as

$$w_1 = 0, \quad \text{and} \quad w_2 \rightarrow \infty.$$

As $(i\omega)^{-1} \rightarrow 0$ the roots may be written as

$$w_1 = A(i\omega)^{-1} + O(\omega^{-2}), \quad \text{and} \quad w_2 = B(i\omega) + C + O(\omega^{-1}). \quad (3.190d)$$

Inserting (3.190d) into (3.190c) we obtain

$$A = \frac{-\tilde{\sigma}}{\varepsilon^{1/2}\alpha_0\tau}, \quad B = \varepsilon^{1/2}\tilde{\sigma}\tau, \quad \text{and} \quad C = \frac{-1}{\alpha_0}[\varepsilon^{1/2}\tilde{\sigma}(\alpha_0 + \alpha_1) - \alpha_0(\tilde{\sigma}^2 - 1)],$$

so the roots become

$$w_1 = \frac{i\omega^{-1}\tilde{\sigma}}{\varepsilon^{1/2}\alpha_0\tau} + O(\omega^{-2}),$$

$$w_2 = i\omega\varepsilon^{1/2}\tilde{\sigma}\tau - \frac{1}{\alpha_0}[\varepsilon^{1/2}\tilde{\sigma}(\alpha_0 + \alpha_1) + \alpha_0(\tilde{\sigma}^2 - 1)] + O(\omega^{-1}).$$

It is clear that $\text{Im } w_1(\omega) > 0$ and $\text{Im } w_2(\omega) > 0$, so w_1 and w_2 are unstable in the high frequency limits.

Now we consider the two special cases.

Case 1: The isothermal constraint viewed as the limit $\tilde{\sigma} \rightarrow 0$

From (3.187)

$$w_{1,2} = \frac{1}{2} \left\{ 1 - \varepsilon^{1/2}\tilde{\sigma} \left[(1 - i\omega\tau) + \frac{1 - i\omega\alpha_1}{1 - i\omega\alpha_0} \right] - \tilde{\sigma}^2 \right\} \pm \frac{1}{2} \left\{ \left[1 - \varepsilon^{1/2}\tilde{\sigma} \left((1 - i\omega\tau) + \frac{1 - i\omega\alpha_1}{1 - i\omega\alpha_0} \right) - \tilde{\sigma}^2 \right]^2 - \frac{4i\omega\tilde{\sigma}^2}{1 - i\omega\alpha_0} \right\}^{1/2}.$$

After expanding and using the binomial expansion we get

$$w_1 = 1 - \varepsilon^{1/2}\tilde{\sigma} \left[(1 - i\omega\tau) + \frac{1 - i\omega\alpha_1}{1 - i\omega\alpha_0} \right] - \tilde{\sigma}^2 - \frac{i\omega\tilde{\sigma}^2}{1 - i\omega\alpha_0} + O(\tilde{\sigma}^3), \quad (3.191)$$

$$w_2 = \frac{i\omega\tilde{\sigma}^2}{1 - i\omega\alpha_0} + O(\tilde{\sigma}^3). \quad (3.192)$$

Equation (3.191) at $\omega = 0$ represents a stable branch starting from the point

$$w = 1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2,$$

and equation (3.192) describes an unstable branch starting from the origin.

Case 2: The purely mechanical constraint viewed as the limit $\tilde{\sigma} \rightarrow \infty$

When $\tilde{\sigma} \rightarrow \infty$ means $\frac{1}{\tilde{\sigma}}$ is small, from (3.187), after expanding and using the binomial expansion, we obtain

$$w_1 = \frac{-i\omega}{1 - i\omega\alpha_0} \left\{ 1 - \varepsilon^{1/2}\tilde{\sigma}^{-1} \left[(1 - i\omega\tau) + \frac{1 - i\omega\alpha_1}{1 - i\omega\alpha_0} \right] \right\} + O(\tilde{\sigma}^{-2}). \quad (3.193)$$

$$w_2 = 1 - \varepsilon^{1/2}\tilde{\sigma} \left[(1 - i\omega\tau) + \frac{1 - i\omega\alpha_1}{1 - i\omega\alpha_0} \right] - \tilde{\sigma}^2 + \frac{i\omega}{1 - i\omega\alpha_0} \left\{ 1 - \varepsilon^{1/2}\tilde{\sigma}^{-1} \left[(1 - i\omega\tau) + \frac{1 - i\omega\alpha_1}{1 - i\omega\alpha_0} \right] \right\} + O(\tilde{\sigma}^{-2}). \quad (3.194)$$

Putting $\alpha_0 = \alpha_1 = 0$ in equations (3.193) and (3.194) we will recover roots of the secular equation in the low and high frequencies in the classical thermoelasticity, see [27, (3.21)-(3.22)].

Numerical results

Figure 3.14 illustrates two longitudinal waves both beginning at the origin one ending at the origin and the other tending to infinity. One of them is stable in the low frequency and unstable in the high frequency and the other branch maintains the instability in the low and high frequency.

Varying the parameters α_0, α_1, τ and ε while changing the magnitude of ω does not have any substantive influence on the stability.

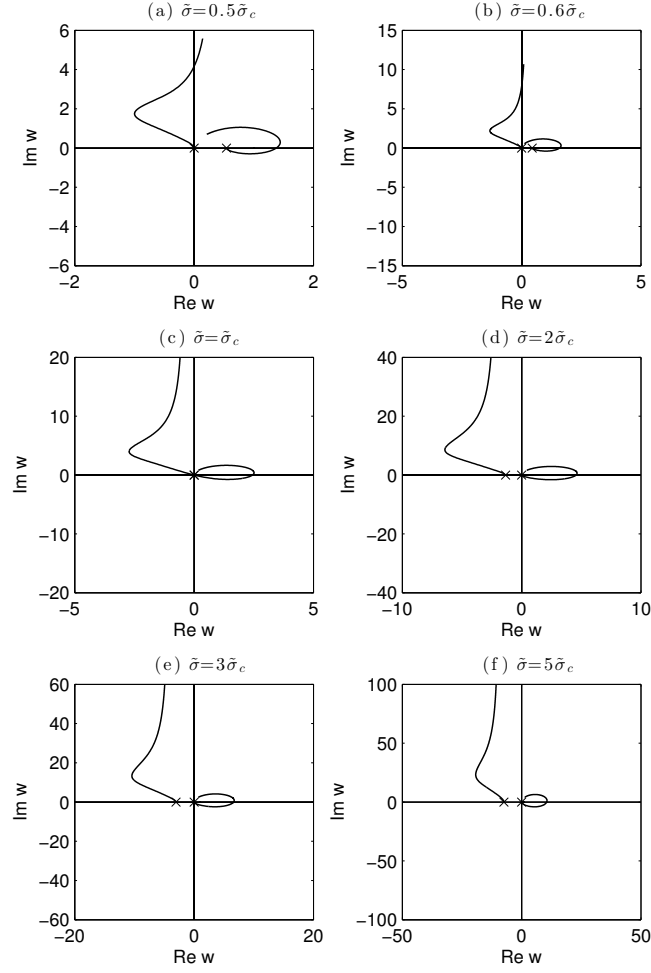


Figure 3.14: The longitudinal squared wave speeds of isotropic thermelastic material for TRDTE+GTE(1) theory with incompressibility at uniform temperature. All plots with same frequencies. For each part ($\varepsilon = 1, \alpha_0 = 0.01, \alpha_1 = 0.02, \tau = 0.1$), (a) $\tilde{\sigma} = 0.1\tilde{\sigma}_c$, (b) $\tilde{\sigma} = 0.3\tilde{\sigma}_c$, (c) $\tilde{\sigma} = \tilde{\sigma}_c$, (d) $\tilde{\sigma} = 2\tilde{\sigma}_c$, (e) $\tilde{\sigma} = 3\tilde{\sigma}_c$, (f) $\tilde{\sigma} = 5\tilde{\sigma}_c$.

Chapter 4

Temperature-rate-dependent thermoelasticity with generalized thermoelasticity: model 2.

Introduction

We are concerned in this chapter with the theory of temperature-rate-dependent thermoelasticity combined with generalized thermoelasticity, due to Ignaczak [25]. We shall indicate this theory by the abbreviation TRDTE+GTE (2). Anisotropic and isotropic thermoelastic materials are considered separately, which are either unconstrained, or constrained by the usual, or the alternative, deformation-temperature constraint. The linearized field equations are given in each case. We will follow the same structure and analysis as we did in the previous chapters. Again, the stability and instability of waves are affected by the values of the relaxation times α_0 , α_1 and τ .

4.1 Unconstrained anisotropic TRDTE+GTE (2)

4.1.1 Basic equations

The basic equations for a homogeneous anisotropic linear thermelastical solids can be described by the following field equations, in the context of Ignaczak's theory, see [25],

$$\left. \begin{aligned} \sigma_{ij} &= \tilde{c}_{ijkl}u_{k,l} - \beta_{ij}\left(1 + \alpha_1\frac{\partial}{\partial t}\right)\theta, \\ \phi &= \rho^{-1}\beta_{ij}n_{i,j} + T^{-1}c\left(1 + \alpha_0\frac{\partial}{\partial t}\right)\theta, \\ \left(1 + \tau\frac{\partial}{\partial t}\right)q_i &= -k_{ij}\theta_{,j}, \end{aligned} \right\} \quad (4.1)$$

in which q_i is heat flux and ϕ is entropy. The balance laws are

$$\sigma_{ij,j} = \rho\ddot{u}_i, \quad -q_{i,i} = \rho T\dot{\phi}. \quad (4.2)$$

4.1.2 The secular equation

In order to get the field equations we have to insert (4.2) into (4.1), similarly to (2.2). Firstly, from (4.1)₁ and (4.2)₁ we get

$$\rho\ddot{u}_i = \tilde{c}_{ijkl}u_{k,lj} - \beta_{ij}\left(1 + \alpha_1\frac{\partial}{\partial t}\right)\theta_{,j}. \quad (4.3)$$

Differentiating (4.1)₂ with respect to time we find that

$$\dot{\phi} = \rho^{-1}\beta_{ij}\dot{u}_{i,j} + T^{-1}c\left(1 + \alpha_0\frac{\partial}{\partial t}\right)\dot{\theta}. \quad (4.4)$$

Multiplying all terms of (4.4) by ρT , we obtain

$$\rho T\dot{\phi} = T\beta_{ij}\dot{u}_{i,j} + \rho c\left(1 + \alpha_0\frac{\partial}{\partial t}\right)\dot{\theta}.$$

Substituting this equation into the balance law (4.2)₂, we get

$$-q_{i,i} = T\beta_{ij}\dot{u}_{i,j} + \rho c\left(1 + \alpha_0\frac{\partial}{\partial t}\right)\dot{\theta}. \quad (4.5)$$

Differentiating (4.1)₃ with respect to x_i we get

$$\left(1 + \tau\frac{\partial}{\partial t}\right)q_{i,i} = -k_{ij}\theta_{,ij}. \quad (4.6)$$

Multiplying (4.5) by $\left(-\left(1+\tau\frac{\partial}{\partial t}\right)\right)$, we obtain

$$\left(1+\tau\frac{\partial}{\partial t}\right)q_{i,i}=-\left(1+\tau\frac{\partial}{\partial t}\right)T\beta_{ij}\dot{u}_{i,j}-\rho c\left(1+\tau\frac{\partial}{\partial t}\right)\left(1+\alpha_0\frac{\partial}{\partial t}\right)\dot{\theta}. \quad (4.7)$$

After expanding the right hand side of (4.7) and inserting (4.6) into (4.7) we obtain

$$k_{ij}\theta_{,ij}=T\beta_{ij}(\dot{u}_{i,j}+\tau\ddot{u}_{i,j})+\rho c(\dot{\theta}+(\alpha_0+\tau)\ddot{\theta}+\alpha_0\tau\ddot{\ddot{\theta}}). \quad (4.8)$$

Hence equations (4.3) and (4.8) represent the field equations of TRDTE and GTE, model 2. Now to get the propagation condition we need to obtain the solution of (4.3) and (4.8) in the form of plane harmonic waves, similar to (2.3). Then all exponential factors are dropped to obtain the propagation conditions. By using the same derivatives as in (2.4a) and the further derivative

$$\ddot{\ddot{\theta}}=(-i\omega)^3\Theta.$$

Inserting the derivatives into (4.3) we get

$$\tilde{c}_{ijkl}(-\omega^2s^2n_l n_j U_k)-i\omega s\beta_{ij}n_j(1-i\omega\alpha_1)\Theta=\rho(-\omega^2U_i).$$

Let us write $U_i=\delta_{ik}U_k$, $\beta_{ij}n_j=b_i$, for convenience, to get

$$\tilde{c}_{ijkl}(-\omega^2s^2n_l n_j +\rho\omega^2\delta_{ik})U_k-i\omega s b_i(1-i\omega\alpha_1)\Theta=0.$$

Dividing by $(-\omega^2s^2)$ we obtain

$$(\tilde{c}_{ijkl}n_l n_j -\rho s^{-2}\delta_{ik})U_k+i(\omega s)^{-1}b_i(1-i\omega\alpha_1)\Theta=0. \quad (4.9)$$

Now substituting the derivatives into (4.8) we get

$$k_{ij}(-\omega^2s^2)n_i n_j\Theta=T\beta_{ij}[\omega^2s n_j U_i+\tau(-i\omega^3s n_j)U_i]+\rho c[-i\omega+(-\omega^2)(\alpha_0+\tau)+\tau\alpha_0 i\omega^3]\Theta.$$

Rearranging the equation after dividing by $(-\omega^2s^2)$, we get

$$T\beta_{ij}n_j s^{-1}(1-i\omega\tau)U_i-(i\omega^{-1}wc(1-i\omega\tau)(1-i\omega\alpha_0)-k_{ij}n_i n_j)\Theta=0. \quad (4.10)$$

Equations (4.9) and (4.10) may be written as

$$\left. \begin{aligned} (\tilde{Q}_{ij}-w\mathbf{1})+b_i(\omega s)^{-1}i(1-i\omega\alpha_1)\Theta &=0, \\ T b_i s^{-1}(1-i\omega\tau)U_i+(k-i\omega^{-1}(1-i\omega\tau)(1-i\omega\alpha_0)wc)\Theta &=0, \end{aligned} \right\} \quad (4.11)$$

where

$$\tilde{Q}_{ij} = \tilde{c}_{ijkl}n_i n_j, \quad b_i = \beta_{ij}n_j, \quad k = k_{ij}n_i n_j.$$

Similar to previous sections, equations (4.11) may be written in matrix form as

$$\begin{bmatrix} \tilde{Q}_{11} - w & \tilde{Q}_{12} & \tilde{Q}_{13} & \bar{\alpha}b_1 \\ \tilde{Q}_{21} & \tilde{Q}_{22} - w & \tilde{Q}_{23} & \bar{\alpha}b_2 \\ \tilde{Q}_{31} & \tilde{Q}_{32} & \tilde{Q}_{33} - w & \bar{\alpha}b_3 \\ \beta b_1 & \beta b_2 & \beta b_3 & \gamma_2 \end{bmatrix} \cdot \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ \Theta \end{bmatrix} = 0,$$

where

$$\beta = Ts^{-1}(1 - i\omega\tau), \quad \gamma_2 = k - i\omega^{-1}(1 - i\omega\tau)(1 - i\omega\alpha_o)cw, \quad \bar{\alpha} = i(\omega s)^{-1}(1 - i\omega\alpha_1). \quad (4.12)$$

The determinant of this equation is

$$\begin{vmatrix} \tilde{Q}_{11} - w & \tilde{Q}_{12} & \tilde{Q}_{13} & \bar{\alpha}b_1 \\ \tilde{Q}_{21} & \tilde{Q}_{22} - w & \tilde{Q}_{23} & \bar{\alpha}b_2 \\ \tilde{Q}_{31} & \tilde{Q}_{32} & \tilde{Q}_{33} - w & \bar{\alpha}b_3 \\ \beta b_1 & \beta b_2 & \beta b_3 & \gamma_2 \end{vmatrix} = 0.$$

Similarly to (2.7), this determinant may be written as follows

$$D \equiv \begin{vmatrix} \tilde{\mathbf{Q}} - w\mathbf{1} & \bar{\alpha}\mathbf{b} + \mathbf{0} \\ \beta\mathbf{b}^T & -\delta + (\gamma_2 + \delta) \end{vmatrix}$$

in which so far δ is an arbitrary quantity. Using properties of determinants to expand by the fourth column we have

$$D \equiv \begin{vmatrix} \tilde{\mathbf{Q}} - w\mathbf{1} & \bar{\alpha}\mathbf{b} \\ \beta\mathbf{b}^T & -\delta \end{vmatrix} + \begin{vmatrix} \tilde{\mathbf{Q}} - w\mathbf{1} & \mathbf{0} \\ \beta\mathbf{b}^T & \gamma + \delta \end{vmatrix}.$$

The first determinant is

$$D_1 = \begin{vmatrix} \tilde{Q}_{11} - w & \tilde{Q}_{12} & \tilde{Q}_{13} & \bar{\alpha}b_1 \\ \tilde{Q}_{21} & \tilde{Q}_{22} - w & \tilde{Q}_{23} & \bar{\alpha}b_2 \\ \tilde{Q}_{31} & \tilde{Q}_{32} & \tilde{Q}_{33} - w & \bar{\alpha}b_3 \\ \beta b_1 & \beta b_2 & \beta b_3 & -\delta \end{vmatrix}.$$

Similarly to the previous sections, remove $\bar{\alpha}\mathbf{b}$ from the fourth column by taking

$$\begin{aligned} \text{row 1} &- \left(\frac{\bar{\alpha}b_1}{-\delta}\right) \text{row 4} \\ \text{row 2} &- \left(\frac{\bar{\alpha}b_2}{-\delta}\right) \text{row 4} \\ \text{row 3} &- \left(\frac{\bar{\alpha}b_3}{-\delta}\right) \text{row 4} \end{aligned}$$

to obtain

$$D_1 = \begin{vmatrix} (\tilde{Q}_{11} - w) - \left(\frac{\bar{\alpha}b_1}{-\delta}\right)\beta b_1 & \tilde{Q}_{12} - \left(\frac{\bar{\alpha}b_1}{-\delta}\right)\beta b_2 & \tilde{Q}_{13} - \left(\frac{\bar{\alpha}b_1}{-\delta}\right)\beta b_3 & 0 \\ \tilde{Q}_{21} - \left(\frac{\bar{\alpha}b_2}{-\delta}\right)\beta b_1 & (\tilde{Q}_{22} - w) - \left(\frac{\bar{\alpha}b_2}{-\delta}\right)\beta b_2 & \tilde{Q}_{23} - \left(\frac{\bar{\alpha}b_2}{-\delta}\right)\beta b_3 & 0 \\ \tilde{Q}_{31} - \left(\frac{\bar{\alpha}b_3}{-\delta}\right)\beta b_1 & \tilde{Q}_{32} - \left(\frac{\bar{\alpha}b_3}{-\delta}\right)\beta b_2 & (\tilde{Q}_{33} - w) - \left(\frac{\bar{\alpha}b_3}{-\delta}\right)\beta b_3 & 0 \\ \beta b_1 & \beta b_2 & \beta b_3 & -\delta \end{vmatrix}$$

and so

$$D_1 = -\delta \det\left\{(\tilde{\mathbf{Q}} - w\mathbf{1}) + \frac{\bar{\alpha}\beta}{\delta}(\mathbf{b} \otimes \mathbf{b})\right\}.$$

From the definitions (2.13) and (4.12) we can get the value of δ as follows

$$\delta = icw\omega^{-1}(1 - i\omega\alpha_1)(1 - i\omega\tau). \quad (4.13)$$

Thus, the first determinant becomes

$$D_1 = -i\omega^{-1}cw(1 - i\omega\alpha_1)(1 - i\omega\tau) \det\{\hat{\mathbf{Q}} - w\mathbf{1}\}. \quad (4.14)$$

The second determinant is

$$D_2 = \begin{vmatrix} (\tilde{Q}_{11} - w) & \tilde{Q}_{12} & \tilde{Q}_{13} & 0 \\ \tilde{Q}_{21} & (\tilde{Q}_{22} - w) & \tilde{Q}_{23} & 0 \\ \tilde{Q}_{31} & \tilde{Q}_{32} & (\tilde{Q}_{33} - w) & 0 \\ \beta b_1 & \beta b_2 & \beta b_3 & \gamma_2 + \delta \end{vmatrix} = (\gamma_2 + \delta) \det\{\tilde{\mathbf{Q}} - w\mathbf{1}\}. \quad (4.15)$$

Since γ_2 is given in (4.12), the second determinant becomes

$$D_2 = [(k - i\omega^{-1}cw(1 - i\omega\alpha_0)(1 - i\omega\tau) + i\omega^{-1}cw(1 - i\omega\alpha_1)(1 - i\omega\tau))] \det\{\tilde{\mathbf{Q}} - w\mathbf{1}\}.$$

After simplifying we get

$$D_2 = [k + wc(1 - i\omega\tau)(\alpha_1 - \alpha_0)] \det\{\tilde{\mathbf{Q}} - w\mathbf{1}\},$$

and so,

$$D \equiv -i\omega^{-1}cw(1-i\omega\alpha_1)(1-i\omega\tau) \det\{\hat{\mathbf{Q}}-w\mathbf{1}\} + (k+cw(1-i\omega\tau)(\alpha_1-\alpha_0)) \det\{\tilde{\mathbf{Q}}-w\mathbf{1}\}. \quad (4.16)$$

Dividing by $(-i\omega^{-1}c(1-i\omega\alpha_1)(1-i\omega\tau))$, we get the secular equation

$$w \det\{\hat{\mathbf{Q}} - w\mathbf{1}\} + \left[\frac{i\omega kc^{-1} + i\omega w(1-i\omega\tau)(\alpha_1-\alpha_0)}{(1-i\omega\alpha_1)(1-i\omega\tau)} \right] \det\{\tilde{\mathbf{Q}} - w\mathbf{1}\} = 0. \quad (4.17)$$

This is the secular equation for unconstrained anisotropic TRDTE+GTE (2) and has not previously appeared in the literature.

To non-dimensionlize this equation we need to use the dimensionless quantities (2.103) and (3.20), then we find that

$$w \det\{\hat{\mathbf{Q}} - w\mathbf{1}\} + \left[\frac{i\omega(1+w(1-i\omega\tau)(\alpha_1-\alpha_0))}{(1-i\omega\alpha_1)(1-i\omega\tau)} \right] \det\{\tilde{\mathbf{Q}} - w\mathbf{1}\} = 0. \quad (4.18)$$

The notations $\hat{\mathbf{Q}}', \tilde{\mathbf{Q}}', w', \omega', \alpha_1', \alpha_0', \tau'$ have been replaced by $\hat{\mathbf{Q}}, \tilde{\mathbf{Q}}, w, \omega, \alpha_1, \alpha_0, \tau$ for convenience. Equation (4.18) may be written as

$$w(w-\hat{q}_1)(w-\hat{q}_2)(w-\hat{q}_3) + \left[\frac{i\omega(1+w(1-i\omega\tau)(\alpha_1-\alpha_0))}{(1-i\omega\alpha_1)(1-i\omega\tau)} \right] (w-\tilde{q}_1)(w-\tilde{q}_2)(w-\tilde{q}_3) = 0, \quad (4.19)$$

in which

$$0 \leq \tilde{q}_1 \leq \hat{q}_1 \leq \tilde{q}_2 \leq \hat{q}_2 \leq \tilde{q}_3 \leq \hat{q}_3, \quad \text{and} \quad \alpha_1 \geq \alpha_0. \quad (4.20)$$

We can rewrite (4.19) as follows

$$\hat{F}(w) + \left[\frac{i\omega(1+w(1-i\omega\tau)(\alpha_1-\alpha_0))}{(1-i\omega\alpha_1)(1-i\omega\tau)} \right] \tilde{G}(w) = 0, \quad (4.21)$$

in which,

$$\hat{F}(w) = w \prod_{i=1}^3 (w - \hat{q}_i), \quad \tilde{G}(w) = \prod_{i=1}^3 (w - \tilde{q}_i).$$

On putting $\tau = 0$ in (4.21) we get the corresponding secular equation (2.27) of TRDTE, as expected.

Low frequency expansions

When $\omega = 0$, the roots of the secular equation (4.21) are the zeros of $\hat{F}(w) : w =$

$\hat{q}_i, i = 0, 1, 2, 3$. Defining $\hat{q}_0 \equiv 0$, Taylor expansions of the roots of equation (4.21) take the form

$$w_i(\omega) = \hat{q}_i + \sum_{n=1}^{\infty} d_n^{(i)}(-i\omega)^n, \quad i = 0, 1, 2, 3. \quad (4.22)$$

The first coefficient, when $i = 0$, is

$$w_0 = \hat{q}_0 + d_1^{(0)}(-i\omega) + O(\omega^2). \quad (4.23)$$

Inserting (4.23) into (4.21) we obtain

$$d_1^{(0)} = \frac{\tilde{q}_1 \tilde{q}_2 \tilde{q}_3}{\hat{q}_1 \hat{q}_2 \hat{q}_3} > 0, \quad (4.24)$$

so

$$w_0 = -i\omega \frac{\tilde{G}(0)}{\hat{F}'(0)} + O(\omega^2). \quad (4.25)$$

It is clear that $\text{Im } w_0 < 0$, so w_0 is stable.

Now, when $i = 1$,

$$w_1 = \hat{q}_1 + d_1^{(1)}(-i\omega) + O(\omega^2). \quad (4.26)$$

Inserting (4.26) into (4.21), and using (4.20) we get

$$d_1^{(1)} = \{1 + \hat{q}_1(\alpha_1 - \alpha_0)\} \frac{(\hat{q}_1 - \tilde{q}_1)(\hat{q}_1 - \tilde{q}_2)(\hat{q}_1 - \tilde{q}_3)}{\hat{q}_1(\hat{q}_1 - \hat{q}_2)(\hat{q}_1 - \hat{q}_3)} > 0. \quad (4.27)$$

Thus,

$$w_1 = \hat{q}_1 - i\omega \{1 + \hat{q}_1(\alpha_1 - \alpha_0)\} \frac{\tilde{G}(\hat{q}_1)}{\hat{F}'(\hat{q}_1)} + O(\omega^2). \quad (4.28)$$

Obviously the stability condition is satisfied because $\text{Im } w_1 < 0$, so it is stable. Similarly, when $i = 2, 3$ we find that

$$w_2 = \hat{q}_2 - i\omega \{1 + \hat{q}_2(\alpha_1 - \alpha_0)\} \frac{\tilde{G}(\hat{q}_2)}{\hat{F}'(\hat{q}_2)} + O(\omega^2), \quad (4.29)$$

$$w_3 = \hat{q}_3 - i\omega \{1 + \hat{q}_3(\alpha_1 - \alpha_0)\} \frac{\tilde{G}(\hat{q}_3)}{\hat{F}'(\hat{q}_3)} + O(\omega^2). \quad (4.30)$$

These two equations represent two stable waves as well. So there are four stable waves in the low frequency limit.

High frequency expansions

When $i\omega \rightarrow \infty$, then $(i\omega)^{-1} \rightarrow 0$. From the secular equation (4.21) we find that

$$w(w - \hat{q}_1)(w - \hat{q}_2)(w - \hat{q}_3) + \kappa(w - \tilde{q}_1)(w - \tilde{q}_2)(w - \tilde{q}_3) = 0, \quad (4.31)$$

where

$$\kappa = \left[\frac{i\omega(1 + w(1 - i\omega\tau)(\alpha_1 - \alpha_0))}{(1 - i\omega\alpha_1)(1 - i\omega\tau)} \right]. \quad (4.32)$$

We can rewrite κ as

$$\kappa = \left[\frac{(i\omega)^{-1} + w((i\omega)^{-1} - \tau)(\alpha_1 - \alpha_0)}{((i\omega)^{-1} - \alpha_1)((i\omega)^{-1} - \tau)} \right].$$

Put $\zeta = 1/\omega$, so $(-i\zeta) = 1/i\omega$. Now rewrite κ in terms of ζ ,

$$\kappa = \left[\frac{(-i\zeta) + w((-i\zeta) - \tau)(\alpha_1 - \alpha_0)}{((-i\zeta) - \alpha_1)((-i\zeta) - \tau)} \right].$$

As $\omega \rightarrow \infty, \zeta \rightarrow 0$, so κ becomes

$$\kappa = \frac{-(\alpha_1 - \alpha_0)}{\alpha_1} w. \quad (4.33)$$

Inserting (4.33) into (4.31), we obtain

$$w(w - \hat{q}_1)(w - \hat{q}_2)(w - \hat{q}_3) - \frac{(\alpha_1 - \alpha_0)}{\alpha_1} w(w - \tilde{q}_1)(w - \tilde{q}_2)(w - \tilde{q}_3) = 0. \quad (4.34)$$

The roots of the secular equation (4.31), when $\omega \rightarrow \infty$, are given by the zeros of $H(w)$

$$H(w) = \hat{F}(w) - w \frac{(\alpha_1 - \alpha_0)}{\alpha_1} \tilde{G}(w), \quad (4.35)$$

where $\hat{F}(w)$ and $\tilde{G}(w)$ are defined earlier in (2.28). Equation (4.35) is a quartic in w , so there are four roots. So that $\bar{q}_i, i = 0, 1, 2, 3$. $\bar{q}_0 \equiv w = 0$ is a root, the other roots are $\bar{q}_1 < \bar{q}_2 < \bar{q}_3$. Now, we have to examine the sign changes of $H(w)$, using the

inequalities (4.20), and equation (4.31), we obtain

$$\begin{aligned}
H(0) &= 0, \\
H(\tilde{q}_1) &= \tilde{q}_1(\tilde{q}_1 - \hat{q}_1)(\tilde{q}_1 - \hat{q}_2)(\tilde{q}_1 - \hat{q}_3) < 0, \\
H(\hat{q}_1) &= -\frac{(\alpha_1 - \alpha_0)}{\alpha_1} \hat{q}_1(\hat{q}_1 - \tilde{q}_1)(\hat{q}_1 - \tilde{q}_2)(\hat{q}_1 - \tilde{q}_3) < 0, \\
H(\tilde{q}_2) &= \tilde{q}_2(\tilde{q}_2 - \hat{q}_1)(\tilde{q}_2 - \hat{q}_2)(\tilde{q}_2 - \hat{q}_3) > 0, \\
H(\hat{q}_2) &= -\frac{(\alpha_1 - \alpha_0)}{\alpha_1} \hat{q}_2(\hat{q}_2 - \tilde{q}_1)(\hat{q}_2 - \tilde{q}_2)(\hat{q}_2 - \tilde{q}_3) > 0, \\
H(\tilde{q}_3) &= \tilde{q}_3(\tilde{q}_3 - \hat{q}_1)(\tilde{q}_3 - \hat{q}_2)(\tilde{q}_3 - \hat{q}_3) < 0, \\
H(\hat{q}_3) &= -\frac{(\alpha_1 - \alpha_0)}{\alpha_1} \hat{q}_3(\hat{q}_3 - \tilde{q}_1)(\hat{q}_3 - \tilde{q}_2)(\hat{q}_3 - \tilde{q}_3) < 0, \\
H(\infty) &= \infty > 0.
\end{aligned} \tag{4.36}$$

It is obvious from (4.36), that the locations of zeros of $H(w)$ satisfy: \bar{q}_1 is between \hat{q}_1 and \tilde{q}_2 , \bar{q}_2 is between \hat{q}_2 and \tilde{q}_3 and \bar{q}_3 is between \hat{q}_3 and ∞ . The roots $\bar{q}_i, i = 1, 2, 3$, therefore satisfy the following inequalities

$$0 < \tilde{q}_1 \leq \hat{q}_1 \leq \bar{q}_1 \leq \tilde{q}_2 \leq \hat{q}_2 \leq \bar{q}_2 \leq \tilde{q}_3 \leq \hat{q}_3 \leq \bar{q}_3. \tag{4.37}$$

On defining the real quartic polynomial

$$\bar{h}(w) = w(w - \bar{q}_1)(w - \bar{q}_2)(w - \bar{q}_3),$$

$\bar{h}(w)$ must be a scalar multiple of $H(w)$ because both have the same four roots, thus

$$H(w) := \left(\frac{\alpha_0}{\alpha_1}\right) \bar{h}(w). \tag{4.38}$$

Now look for roots when $\omega \rightarrow \infty$, from (4.32), putting $\omega = \frac{1}{\zeta} \Rightarrow i\omega = -(i\zeta)^{-1}$, so,

$$\kappa = \frac{-(i\zeta)^{-1}(1 + w(1 + (i\zeta)^{-1}\tau)(\alpha_1 - \alpha_0))}{(i\zeta)^{-2}((i\zeta) + \alpha_1)((i\zeta) + \tau)}.$$

Multiplying numerator and denominator of κ by $(i\zeta)^2$ we get

$$\kappa = \frac{-(i\zeta)(1 + w(1 + (i\zeta)^{-1}\tau)(\alpha_1 - \alpha_0))}{\alpha_1\tau(1 + \frac{i\zeta}{\alpha_1})(1 + \frac{i\zeta}{\tau})}.$$

After expanding and simplifying we get

$$\kappa = -w(1 - \frac{\alpha_0}{\alpha_1}) - \frac{i\zeta}{\alpha_1\tau} \{1 - w\tau(1 - \alpha_0/\alpha_1)\} \quad (4.39)$$

and so (4.21) may be written as

$$\hat{F}(w) + \kappa(\omega, w)\tilde{G}(w) = 0. \quad (4.40)$$

Inserting (4.39) into (4.40) we get

$$\left(\hat{F}(w) - w(1 - \frac{\alpha_0}{\alpha_1})\tilde{G}(w)\right) - \frac{i\zeta}{\alpha_1\tau} \{1 - w\tau(1 - \alpha_0/\alpha_1)\}\tilde{G}(w) = 0. \quad (4.41)$$

But from (4.35) and (4.38), equation (4.41) may be written as

$$\bar{h}(w) - \frac{i\zeta}{\alpha_0\tau} \{1 - w\tau(1 - \alpha_0/\alpha_1)\}\tilde{G}(w) = 0. \quad (4.42)$$

Taylor expansions of the roots of (4.42) take the form

$$w_i(\zeta) = \bar{q}_i + \sum_{n=1}^{\infty} d_n^{(i)}(i\zeta)^n, \quad i = 1, 2, 3, 4. \quad (4.43)$$

The first coefficient, when $i = 0$, is

$$w_0(\zeta) = \bar{q}_0 + d_1^{(0)}(i\zeta) + O(\zeta^2). \quad (4.44)$$

Substituting (4.44) into (4.42), taking $\bar{q}_0 \equiv 0$ and $\zeta \rightarrow 0$, we get

$$d_1^{(0)} = \left(\frac{1}{\alpha_0\tau}\right) \frac{\tilde{q}_1\tilde{q}_2\tilde{q}_3}{\bar{q}_1\bar{q}_2\bar{q}_3} > 0. \quad (4.45)$$

Inserting (4.45) into (4.44) we get

$$w_0(\zeta) = \left(\frac{i\zeta}{\alpha_0\tau}\right) \frac{\tilde{G}(0)}{\bar{h}'(0)} + O(\zeta^2). \quad (4.46)$$

As we see here, $\text{Im } w_0 > 0$, so w_0 is unstable. Now when $i = 1, n = 1$, (4.43) becomes

$$w_1(\zeta) = \bar{q}_1 + d_1^{(1)}(i\zeta) + O(\zeta^2). \quad (4.47)$$

Again, inserting (4.47) into (4.44) we get

$$d_1^{(1)} = \frac{1}{\alpha_0\tau} \left(1 - \bar{q}_1\tau(1 - \alpha_0/\alpha_1)\right) \frac{(\bar{q}_1 - \tilde{q}_1)(\bar{q}_1 - \tilde{q}_2)(\bar{q}_1 - \tilde{q}_3)}{\bar{q}_1(\bar{q}_1 - \bar{q}_2)(\bar{q}_1 - \bar{q}_3)} > 0. \quad (4.48)$$

Substituting (4.48) into (4.47) we get

$$w_1(\zeta) = \bar{q}_1 + \frac{i\zeta}{\alpha_0\tau} \left(1 - \bar{q}_1\tau(1 - \alpha_0/\alpha_1)\right) \frac{\tilde{G}(\bar{q}_1)}{\bar{h}'(\bar{q}_1)} + O(\zeta^2). \quad (4.49)$$

The stability condition depends on τ ; if τ is very large then the condition is satisfied here, so w_1 is stable, but if τ is very small then w_1 is unstable.

Similarly, when $i = 2$ and 3 , we find that

$$w_2(\zeta) = \bar{q}_2 + \frac{i\zeta}{\alpha_0\tau} \left(1 - \bar{q}_2\tau(1 - \alpha_0/\alpha_1)\right) \frac{\tilde{G}(\bar{q}_2)}{\bar{h}'(\bar{q}_2)} + O(\zeta^2), \quad (4.50)$$

$$w_3(\zeta) = \bar{q}_3 + \frac{i\zeta}{\alpha_0\tau} \left(1 - \bar{q}_3\tau(1 - \alpha_0/\alpha_1)\right) \frac{\tilde{G}(\bar{q}_3)}{\bar{h}'(\bar{q}_3)} + O(\zeta^2). \quad (4.51)$$

Again, these branches could be stable when τ is very large, and unstable if τ is very small. We conclude that

$$w_i, \text{ is stable for } 1 - \bar{q}_i\tau(1 - \alpha_0/\alpha_1) < 0, \text{ for } i = 1, 2, 3.$$

We can say that

$$\begin{aligned} w_1 & \text{ stable for, } \tau > \tau_1 \equiv \frac{1}{\bar{q}_1(1 - \alpha_0/\alpha_1)}, \\ w_2 & \text{ stable for, } \tau > \tau_2 \equiv \frac{1}{\bar{q}_2(1 - \alpha_0/\alpha_1)}, \\ w_3 & \text{ stable for, } \tau > \tau_3 \equiv \frac{1}{\bar{q}_3(1 - \alpha_0/\alpha_1)}. \end{aligned}$$

Now

$$\begin{aligned} \tau > \tau_1 > \tau_2 > \tau_3, & \text{ all three branches are stable.} \\ \tau_1 > \tau > \tau_2 > \tau_3, & \bar{q}_1 \text{ is unstable but, } \bar{q}_2, \bar{q}_3 \text{ are stable.} \\ \tau_1 > \tau_2 > \tau > \tau_3, & \bar{q}_1, \bar{q}_2, \text{ are unstable but, } \bar{q}_3 \text{ is stable.} \\ \tau_1 > \tau_2 > \tau_3 > \tau, & \text{ all three branches are unstable.} \end{aligned}$$

Numerical results

Figure 4.1 illustrates that, the stability condition depends on values of τ , α_1 and α_0 , demonstrating the effect of τ increasing. When τ is very small but not zero the branches become unstable. If τ is very big there are three stable waves but one mode be always unstable in high frequency limit.

In each sub-figure we select the same values of $\tilde{q}_i, i = 1, 2, 3$, and $\hat{q}_i, i = 1, 2, 3$. Low frequency limits are marked with a \times and high frequency limits with \bullet . It is clear that all branches lie in the lower complex w -plane and only the first subfigure (a)

shows that all branches satisfy the stability condition (2.24) and it is corresponding to subfigure (b) of Figure 2.1 in TRDTE. .

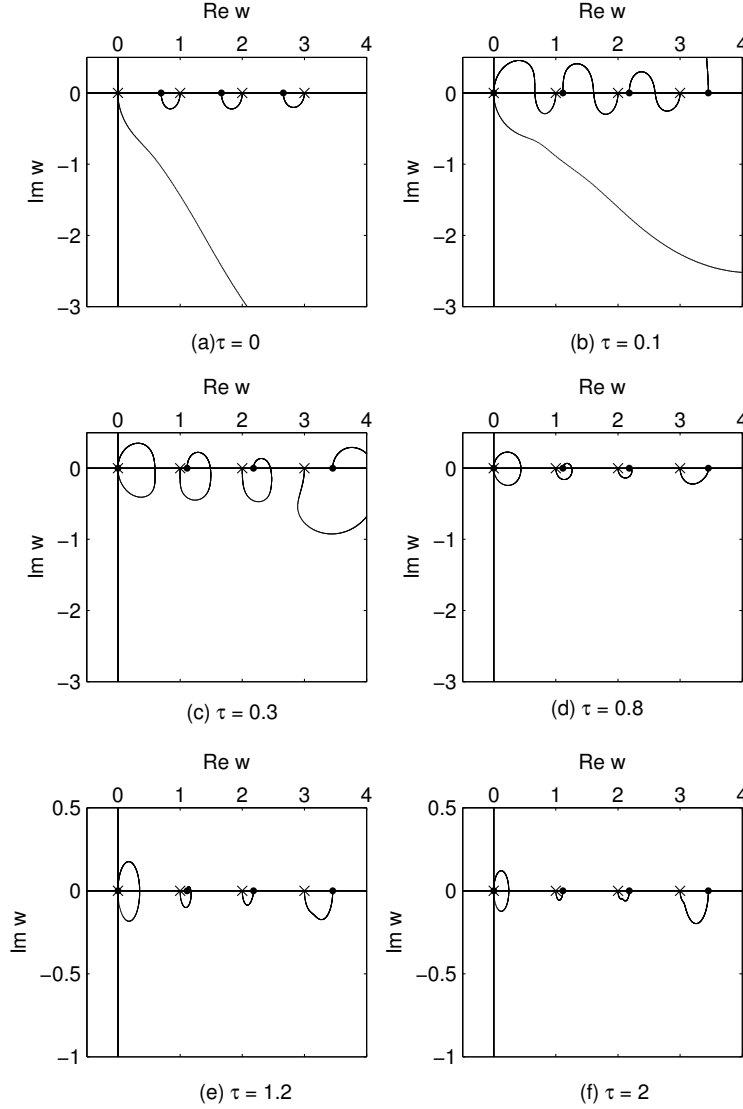


Figure 4.1: The four branches of the secular equation for unconstrained anisotropic thermelastic material for TRDTE + GTE (2). For each part, $\alpha_0 = 0.1$, $\alpha_1 = 0.2$, $\tilde{q}_1 = 1$, $\tilde{q}_2 = 2$, $\tilde{q}_3 = 3$, $\hat{q}_1 = 0.75$, $\hat{q}_2 = 1.75$, $\hat{q}_3 = 2.75$.

4.2 Unconstrained isotropic TRDTE+GTE (2)

In order to get the field equations of an isotropic unconstrained thermoelastic material in the context of Ignaczak's theory we should follow the same steps as in Sections 2.2 and 3.2. Thus, the field equations are

$$\left. \begin{aligned} (\tilde{\lambda} + \tilde{\mu})u_{j,ij} + \tilde{\mu}u_{i,jj} - \beta(\theta + \alpha_1\dot{\theta})_{,i} &= \rho\ddot{u}_i, \\ k\theta_{,ii} - T\beta\left(1 + \tau\frac{\partial}{\partial t}\right)\dot{u}_{j,j} - \rho c(1 + (\alpha_0 + \tau)\frac{\partial}{\partial t} + \alpha_0\tau\frac{\partial^2}{\partial t^2})\dot{\theta} &= 0. \end{aligned} \right\} \quad (4.52)$$

4.2.1 The secular equation

Now we are seeking the solutions of (4.52) in the form of plane harmonic waves (3.2) and follow the same steps in the previous chapters to obtain

$$\left. \begin{aligned} ((\tilde{\lambda} + \tilde{\mu})n_i n_j + (\tilde{\mu} - \rho s^{-2})\delta_{ij})U_j + i\beta(\omega s)^{-1}n_i(1 - i\omega\alpha_1)\Theta &= 0, \\ T\beta\omega s n_j(1 - i\omega\tau)U_j + (k\omega s^2 - i\rho c(1 - i\omega\alpha_0)(1 - i\omega\tau))\Theta &= 0. \end{aligned} \right\} \quad (4.53)$$

Eliminating Θ between (4.53) we get

$$\left\{ (\tilde{\mu} - w)\delta_{ij} + (\tilde{\lambda} + \tilde{\mu})n_i n_j + \frac{wT\beta^2(1 - i\omega\alpha_1)(1 - i\omega\tau)n_i n_j}{\rho c(w(1 - i\omega(\alpha_0 + \tau) - \omega^2\alpha_0\tau) + i\omega k/c)} \right\} U_j = 0, \quad (4.54)$$

so that there exist non-zero amplitudes satisfying (4.54) if and only if

$$\det \left\{ (\tilde{\mu} - w)\mathbf{1} + \left[(\tilde{\lambda} + \tilde{\mu}) + \frac{wT\beta^2(1 - i\omega\alpha_1)(1 - i\omega\tau)}{\rho c(w(1 - i\omega(\alpha_0 + \tau) - \omega^2\alpha_0\tau) + i\omega k/c)} \right] \mathbf{n} \otimes \mathbf{n} \right\} = 0. \quad (4.55)$$

Using the dimensionless quantities (2.57) to non-dimensionlize (4.55) we get

$$\det \left\{ (\tilde{\mu}' - w')\mathbf{1} + \left(\tilde{\lambda}' + \tilde{\mu}' + \frac{\varepsilon w'(1 - i\omega'\alpha'_1)(1 - i\omega'\tau')}{w'(1 - i\omega'(\alpha'_0 + \tau') - \omega'^2\alpha'_0\tau') + i\omega'} \right) \mathbf{n} \otimes \mathbf{n} \right\} = 0. \quad (4.56)$$

Using again the standard identity (2.60), dropping the dashes for convenience, we get

$$(w - \tilde{\mu})^2[w^2(1 - i\omega\alpha_0)(1 - i\omega\tau) - w(1 - i\omega\alpha_0)(1 - i\omega\tau)(\tilde{\lambda} + 2\tilde{\mu}) + \varepsilon(1 - i\omega\alpha_1)(1 - i\omega\tau) - i\omega] - (\tilde{\lambda} + 2\tilde{\mu})i\omega = 0. \quad (4.57)$$

This is the secular equation for unconstrained isotropic TRDTE+GTE (2) and has not previously appeared in the literature.

Since $(\tilde{\lambda} + 2\tilde{\mu}) = 1$, the longitudinal waves are represented by the following quadratic equation,

$$w^2(1 - i\omega\alpha_0)(1 - i\omega\tau) - w[(1 - i\omega\alpha_0)(1 - i\omega\tau) + \varepsilon(1 - i\omega\alpha_1)(1 - i\omega\tau) - i\omega] - i\omega = 0. \quad (4.58)$$

On putting $\tau = 0$ in (4.58) we get the corresponding secular equation (2.70) of TRDTE, as expected. This equation may be written as

$$w^2 - w\left[1 + \varepsilon\left(\frac{1 - i\omega\alpha_1}{1 - i\omega\alpha_0}\right) - \frac{i\omega}{(1 - i\omega\alpha_0)(1 - i\omega\tau)}\right] - \frac{i\omega}{(1 - i\omega\alpha_0)(1 - i\omega\tau)} = 0,$$

with roots

$$w_{1,2} = \frac{1}{2}\left\{z_3 \pm \left((z_3)^2 + \frac{4i\omega}{(1 - i\omega\alpha_0)(1 - i\omega\tau)}\right)^{\frac{1}{2}}\right\}, \quad (4.59)$$

where

$$z_3 = 1 + \varepsilon\left(\frac{1 - i\omega\alpha_1}{1 - i\omega\alpha_0}\right) - \frac{i\omega}{\kappa_1},$$

and

$$\kappa_1 = (1 - i\omega\alpha_0)(1 - i\omega\tau).$$

In Figure 4.2 the roots (4.59) are plotted in the complex w -plane for various values of $\varepsilon \geq 0$. In the uncoupled case, when $\varepsilon = 0$, the roots become

$$w_1 = 1, \quad w_2 = \frac{-i\omega}{\kappa_1}, \quad (4.60)$$

see sub-figure 4.2 (a). Equation (4.60)₁ corresponds to the unattenuated, non-dispersive longitudinal wave and is called an elastic mode. Equation (4.60)₂ corresponds to the pure diffusion equation and is called a diffusive mode.

Low frequency expansions

Now returning to the general case when $\varepsilon \geq 0$ for low frequencies at $\omega = 0$ the roots (4.59) become

$$w_1 = 1 + \varepsilon, \quad w_2 = 0. \quad (4.60a)$$

But as $\omega \rightarrow 0$, the roots may be written as

$$w_1 = (1 + \varepsilon) + A(i\omega) + O(\omega^2), \quad w_2 = B(i\omega) + O(\omega^2). \quad (4.60b)$$

Inserting (4.60b) into (4.58) we get

$$A = -\varepsilon[\alpha_1 - \alpha_0 + \frac{1}{1+\varepsilon}], \quad \text{and} \quad B = \frac{-1}{1+\varepsilon}.$$

Thus, the roots become

$$w_1 = 1 + \varepsilon - i\omega\varepsilon[\alpha_1 - \alpha_0 + \frac{1}{1+\varepsilon}] + O(\omega^2), \quad (4.61)$$

$$w_2 = \frac{-i\omega}{1+\varepsilon} + O(\omega^2). \quad (4.62)$$

Both branches are stable in the low frequency limits.

High frequency expansions

In the high frequencies as $\omega \rightarrow \infty$, $\omega^{-1} \rightarrow 0$ The secular equation (4.58) may be written as

$$\begin{aligned} & w^2[1 - i\omega(\alpha_0 + \tau) + (i\omega)^2\alpha_0\tau] \\ & - w[(1 - i\omega(\alpha_0 + \tau) + (i\omega)^2\alpha_0\tau) + \varepsilon(1 - i\omega(\alpha_0 + \tau) + (i\omega)^2\alpha_1\tau) - i\omega] - i\omega = 0. \end{aligned} \quad (4.60c)$$

After dividing by $(i\omega)^2$ equation (4.60c) becomes

$$\begin{aligned} & w^2[(i\omega)^{-2} - (i\omega)^{-1}(\alpha_0 + \tau) + \alpha_0\tau] \\ & - w\left\{[(i\omega)^{-2} - (i\omega)^{-1}(\alpha_0 + \tau) + \alpha_0\tau] + \varepsilon[(i\omega)^{-2} - i\omega(\alpha_0 + \tau) + \alpha_1\tau] - (i\omega)^{-1}\right\} - (i\omega)^{-1} = 0. \end{aligned} \quad (4.60d)$$

Putting $(i\omega)^{-1} = 0$ into (4.60d) we get

$$w^2(\alpha_0\tau) - w(\alpha_0\tau + \varepsilon\alpha_1\tau) = 0,$$

and the roots are

$$w_1 = 1 + \varepsilon\frac{\alpha_1}{\alpha_0}, \quad \text{and} \quad w_2 = 0. \quad (4.60e)$$

Now we need to get the roots of the secular equation as $(i\omega)^{-1} \rightarrow 0$. The roots may be written in the following forms

$$w_1 = 1 + \varepsilon\frac{\alpha_1}{\alpha_0} + A(i\omega)^{-1} + O(\omega^{-2}), \quad \text{and} \quad w_2 = B(i\omega)^{-1} + O(\omega^{-2}). \quad (4.60f)$$

Substituting (4.60f) into (4.60d) we obtain

$$A = \frac{\varepsilon}{\alpha_0^2} \left[\alpha_1 - \alpha_0 - \frac{\alpha_0}{\varepsilon\tau} \right] + \frac{1}{\tau(\alpha_0 + \varepsilon\alpha_1)},$$

and

$$B = \frac{-1}{\tau(\alpha_0 + \varepsilon\alpha_1)}.$$

Thus, the roots are

$$w_1 = 1 + \varepsilon \frac{\alpha_1}{\alpha_0} - i\omega^{-1} \left\{ \frac{\varepsilon}{\alpha_0^2} \left[\alpha_1 - \alpha_0 - \frac{\alpha_0}{\varepsilon\tau} \right] + \frac{1}{\tau(\alpha_0 + \varepsilon\alpha_1)} \right\} + O(\omega^{-2}), \quad (4.63)$$

and

$$w_2 = \frac{i\omega^{-1}}{\tau(\alpha_0 + \varepsilon\alpha_1)} + O(\omega^{-1}). \quad (4.64)$$

We cannot tell about the sign of $\text{Im } w_1(\omega)$ because it depends on the relative values of the quantities occurring. But it is clear that $\text{Im } w_2(\omega) > 0$, so w_2 is unstable.

Numerical results

It can be seen that in each part of Figure 4.2 for $\varepsilon \geq 0$ the roots (4.60a) are elastic in character and stable for low frequency and are marked \times in each part of the figure. In the high frequency limit, (4.60e) are also elastic but unstable and are marked \circ in each part of the figure .

Varying the parameters α_1, α_0 and τ while changing the magnitude of ω does not have any substantive influence on the stability.

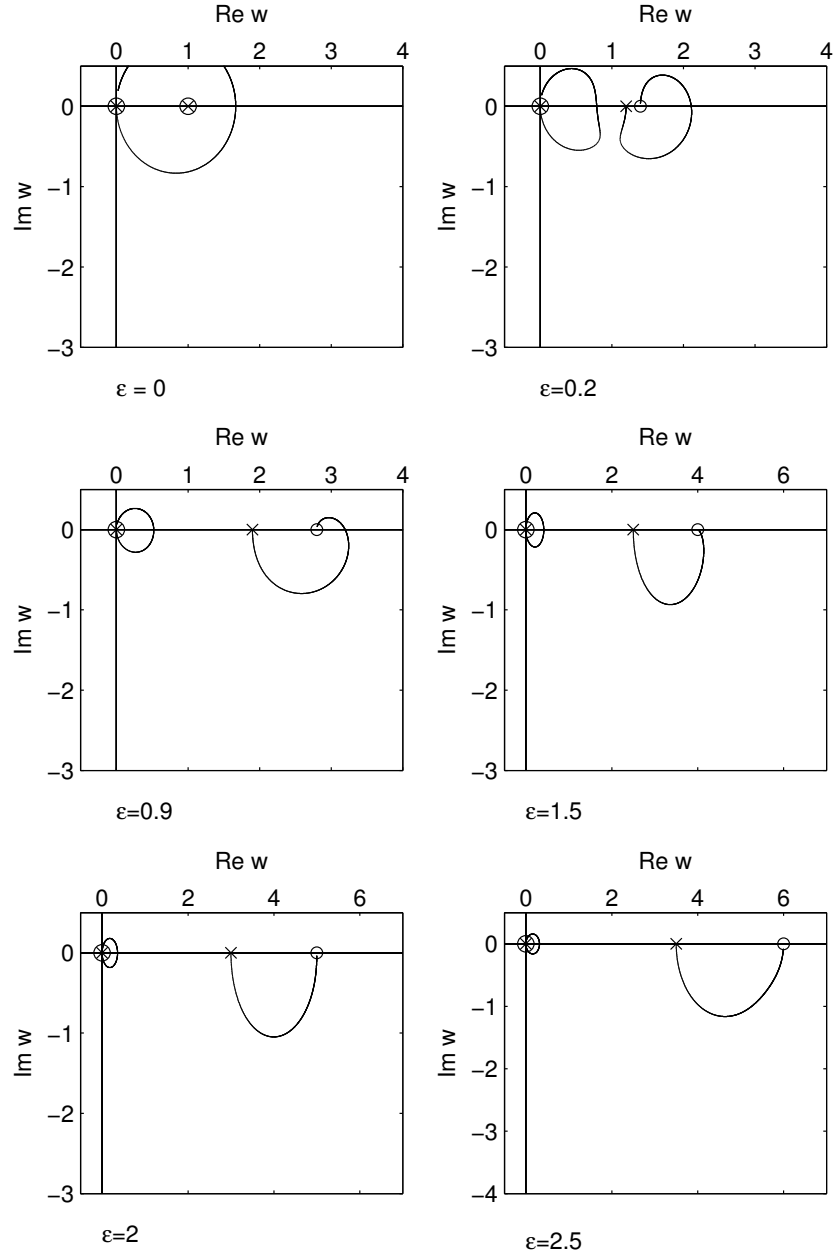


Figure 4.2: The longitudinal squared wave speeds of unconstrained isotropic TRDTE + GTE (2). For each part, $\alpha_0 = 0.1$, $\alpha_1 = 0.2$, $\tau = 0.5$.

4.3 Constrained anisotropic TRDTE+GTE (2)

4.3.1 Usual form of deformation-temperature constraint

Similarly to Sections 2.3.1 and 3.3.1, the linearized form of the constraint connecting the deformation and the temperature takes the form

$$\tilde{N}_{qp}u_{p,q} - \alpha\theta = 0. \quad (4.65)$$

The constitutive equations for a deformation-temperature constrained material are

$$\left. \begin{aligned} \sigma_{ij} &= \tilde{c}_{ijkl}u_{k,l} - \beta_{ij}\left(1 + \alpha_1\frac{\partial}{\partial t}\right)\theta + \tilde{\eta}\tilde{N}_{ij}, \\ \phi &= \rho^{-1}\beta_{ij}u_{i,j} + T^{-1}c\left(1 + \alpha_0\frac{\partial}{\partial t}\right)\theta + \rho^{-1}\alpha\tilde{\eta}. \end{aligned} \right\} \quad (4.66)$$

In order to get the field equations, arguing similarly to (2.82), we find that on inserting (4.66)₁ into the balance equations (4.2)₁, we get

$$\tilde{c}_{ijkl}u_{k,l,j} - \beta_{ij}\left(1 + \alpha_1\frac{\partial}{\partial t}\right)\theta_{,j} + \tilde{N}_{ij}\tilde{\eta}_{,j} = \rho\ddot{u}_i. \quad (4.67)$$

Equation (4.67) is similar to (2.83). From the modified Fourier law (4.1)₃ and (4.2)₂, we obtain

$$\left(1 + \tau\frac{\partial}{\partial t}\right)q_{i,i} = -k_{ij}\theta_{,ij}. \quad (4.68)$$

Differentiate (4.66)₂ with respect to time after multiplying by ρT to get

$$\rho T\dot{\phi} = T\beta_{ij}\dot{u}_{i,j} + \rho c\left(1 + \alpha_0\frac{\partial}{\partial t}\right)\dot{\theta} + T\alpha\dot{\tilde{\eta}}.$$

Inserting into (4.2)₂ after multiplying by $\left(1 + \tau\frac{\partial}{\partial t}\right)$, we get

$$\left(1 + \tau\frac{\partial}{\partial t}\right)q_{i,i} = -[T\beta_{ij}(\dot{u}_{i,j} + \tau\ddot{u}_{i,j}) + \rho c(\dot{\theta} + (\alpha_0 + \tau)\ddot{\theta} + \alpha_0\tau\ddot{\ddot{\theta}}) + T\alpha(\dot{\tilde{\eta}} + \tau\ddot{\tilde{\eta}})]. \quad (4.69)$$

Substituting (4.68) into (4.69) we obtain

$$T\beta_{pq}(\dot{u}_{p,q} + \tau\ddot{u}_{p,q}) + \rho c(\dot{\theta} + (\alpha_0 + \tau)\ddot{\theta} + \alpha_0\tau\ddot{\ddot{\theta}}) + T\alpha(\dot{\tilde{\eta}} + \tau\ddot{\tilde{\eta}}) - k_{pq}\theta_{,pq} = 0. \quad (4.70)$$

The secular equation

Now we are looking for the solutions of (4.65), (4.67) and (4.70) by substituting them

in the same form of plane harmonic waves (2.86). We already have most of derivatives in (2.4a), and now just observe that

$$\dot{u}_{p,q} = \omega^2 s n_q U_p e^\chi, \quad \ddot{u}_{p,q} = -i\omega^3 s n_q U_p e^\chi, \quad \ddot{\theta} = i\omega^3 \Theta e^\chi, \quad \ddot{\eta} = -\omega^2 \tilde{H} e^\chi,$$

where

$$\chi = i\omega(s\mathbf{n} \cdot \mathbf{x} - t).$$

Inserting all the derivatives into (4.65), (4.67) and (4.70) we get

$$\tilde{c}_{ijkl}(-(\omega s)^2) n_l n_j U_k - \beta_{ij}(i\omega s n_j(1 - i\omega\alpha_1)\Theta) + \tilde{N}_{ij}(i\omega s n_j \tilde{H}) = \rho(-\omega^2)U_i. \quad (4.71)$$

Dividing by $(-\omega s)^2$, we get

$$\tilde{c}_{ijkl} n_l n_j U_k + \beta_{ij} n_j i(\omega s)^{-1}(1 - i\omega\alpha_1)\Theta - i(\omega s)^{-1} \tilde{N}_{ij} n_j \tilde{H} = \rho s^{-2} U_i. \quad (4.72)$$

Rearranging the equation we obtain

$$(\tilde{Q}_{ik} - \rho s^{-2} \delta_{ik}) U_k + i(\omega s)^{-1} [b_i(1 - i\omega\alpha_1)\Theta - \tilde{c}_i \tilde{H}] = 0. \quad (4.73)$$

This equation is similar to (2.89) in TRDTE and (3.82) in TRDTE + GTE, model 1. Equation (4.70), after inserting the derivatives, becomes

$$T\beta_{pq}[(\omega^2 s n_q U_p) + \tau(-i\omega^3 s n_q U_p)] + \rho c(-i\omega + (\alpha_0 + \tau)(-\omega^2) + \alpha_0 \tau(i\omega^3))\Theta + \\ T\alpha(-i\omega + \tau(-\omega^2))\tilde{H} - k_{ij}(i\omega s)^2 n_i n_j \Theta = 0.$$

Rearranging the equation after dividing by ω we obtain

$$\omega s T b_p (1 - i\omega\tau) U_p - iT\alpha(1 - i\omega\tau)\tilde{H} + (\omega s^2 k - i\rho c(1 - i(\alpha_0 + \tau)\omega - \alpha_0 \tau \omega^2))\Theta = 0, \quad (4.74)$$

where k and b_p have been mentioned earlier in the previous theories. By putting $\tau = 0$ in this equation we will get the same equation (2.91) as in TRDTE. Equation (4.65), after inserting derivatives, becomes

$$i\omega s \tilde{c}_p U_p - \alpha \Theta = 0, \quad (4.75)$$

where \tilde{c}_p is defined formerly in TRDTE (for the anisotropic constrained case). We now eliminate Θ and \tilde{H} between equations (4.73), (4.74) and (4.75). Equation (4.75) may be rewritten as

$$\Theta = \alpha^{-1} i\omega s \tilde{c}_p U_p. \quad (4.76)$$

Inserting (4.76) into (4.74), we get

$$\tilde{H} = -i\omega s\alpha^{-1}b_p U_p + (\omega s^2 k - i\rho c(1 - i(\alpha_0 + \tau)\omega - \alpha_0\tau\omega^2))(\alpha^2 T)^{-1}(1 - i\omega\tau)^{-1}\omega s\tilde{c}_p U_p. \quad (4.77)$$

Substituting (4.76) and (4.77) into (4.73) gives

$$(\tilde{Q}_{ik} - \rho s^{-2}\delta_{ik})U_k + i(\omega s)^{-1}[b_i(1 - i\omega\alpha_1)(\alpha^{-1}i\omega s\tilde{c}_p U_p) - \tilde{c}_i[-i\omega s\alpha^{-1}b_p U_p + (\omega s^2 k - i\rho_e c(1 - i(\alpha_0 + \tau)\omega - \alpha_0\tau\omega^2))(\alpha^2 T)^{-1}(1 - i\omega\tau)^{-1}\omega s\tilde{c}_p U_p] = 0.$$

Rearranging this equation, we get

$$\{\tilde{Q}_{ip} - \alpha^{-1}(b_i\tilde{c}_p(1 - i\omega\alpha_1) + \tilde{c}_i b_p) - (\alpha^2 T)^{-1}(i\omega s^2 k + \rho c(1 - i(\alpha_0 + \tau)\omega - \alpha_0\tau\omega^2))\tilde{c}_i\tilde{c}_p - \rho s^{-2}\delta_{ip}\}U_p = 0. \quad (4.78)$$

The non-zero amplitudes satisfy (4.73), (4.74) and (4.75) if and only if

$$\det\{\tilde{\mathbf{Q}} - \alpha^{-1}(\mathbf{b} \otimes \tilde{\mathbf{c}} + \tilde{\mathbf{c}} \otimes \mathbf{b}) - (\alpha^2 T)^{-1}(\rho c + i\omega s^2 k)\tilde{\mathbf{c}} \otimes \tilde{\mathbf{c}} + \alpha^{-1}i\omega\alpha_1\mathbf{b} \otimes \tilde{\mathbf{c}} + i(\alpha^2 T)^{-1}\rho c\omega((\alpha_0 + \tau) - i\omega\alpha_0\tau)\tilde{\mathbf{c}} \otimes \tilde{\mathbf{c}} - \rho s^{-2}\mathbf{1}\} = 0. \quad (4.79)$$

Defining

$$\tilde{\mathbf{P}} := \tilde{\mathbf{Q}} - \alpha^{-1}(\mathbf{b} \otimes \tilde{\mathbf{c}} + \tilde{\mathbf{c}} \otimes \mathbf{b}) - (\alpha^2 T)^{-1}(\rho c)\tilde{\mathbf{c}} \otimes \tilde{\mathbf{c}}, \quad (4.80)$$

we can rewrite (4.79) in terms of the tensor $\tilde{\mathbf{P}}$ as

$$\det\left\{(\tilde{\mathbf{P}} - w\mathbf{1}) + \alpha^{-1}i\omega\alpha_1\mathbf{b} \otimes \tilde{\mathbf{c}} - \frac{i\omega}{\alpha^2 T}(s^2 k - \rho c((\alpha_0 + \tau) - i\omega\alpha_0\tau))\tilde{\mathbf{c}} \otimes \tilde{\mathbf{c}}\right\} = 0. \quad (4.81)$$

Equation (4.81) may be rewritten as

$$\det\{(\tilde{\mathbf{P}} - w\mathbf{1}) + [\alpha^{-1}i\omega\alpha_1\mathbf{b} - \frac{i\omega}{\alpha^2 T}(s^2 k - \rho c((\alpha_0 + \tau) - i\omega\alpha_0\tau))]\tilde{\mathbf{c}} \otimes \tilde{\mathbf{c}}\} = 0. \quad (4.82)$$

Using the standard identity (2.60), we get

$$\det(\tilde{\mathbf{P}} - w\mathbf{1}) + [\alpha^{-1}i\omega\alpha_1\mathbf{b} - \frac{i\omega}{\alpha^2 T}(s^2 k - \rho c((\alpha_0 + \tau) - i\omega\alpha_0\tau))]\tilde{\mathbf{c}} \cdot (\tilde{\mathbf{P}} - w\mathbf{1})^{\text{adj}}\tilde{\mathbf{c}} = 0. \quad (4.83)$$

Now we need to rewrite $\tilde{\mathbf{P}}$ in terms of definitions (2.110a) and (2.110b) to get the same equation (2.111) as before. In dimensionless form, if we use the dimensionless quantities (2.103) and (3.20) then we will obtain the same equation (2.112) as before.

Also, the secular equation (4.83) may be written in terms of definitions (2.110a) and (2.110b) as

$$\det(\tilde{\mathbf{P}} - w\mathbf{1}) + \left[\alpha^{-1} i\omega \alpha_1 \beta - \frac{i\omega}{\alpha^2 T} \left(s^2 k - \rho c ((\alpha_0 + \tau) - i\omega \alpha_0 \tau) \right) \right] \mathbf{n} \cdot (\tilde{\mathbf{P}} - w\mathbf{1})^{\text{adj}} \mathbf{n} = 0. \quad (4.84)$$

This is the secular equation for anisotropic TRDTE+GTE (2) which is constrained by the usual deformation temperature constraint and has not previously appeared in the literature.

Now, the dimensionless form of the secular equation (4.84) is

$$w \det(w\mathbf{1} - \tilde{\mathbf{P}}) - i\omega \tilde{\sigma} \left[w \left(\varepsilon^{\frac{1}{2}} \alpha_1 + \tilde{\sigma} (\alpha_0 + \tau (1 - i\omega \alpha_0)) \right) - \tilde{\sigma} \right] \mathbf{n} \cdot (w\mathbf{1} - \tilde{\mathbf{P}})^{\text{adj}} \mathbf{n} = 0, \quad (4.85)$$

dropping the dashes for convenience. We will follow exactly the same steps as in Sections 2.3.1 and 3.3.1 of the usual deformation temperature constraint, to get the final form of the secular equation here as

$$\tilde{F}(w) - i\omega \tilde{\sigma} \left[w \left(\varepsilon^{\frac{1}{2}} \alpha_1 + \tilde{\sigma} (\alpha_0 + \tau (1 - i\omega \alpha_0)) \right) - \tilde{\sigma} \right] \tilde{G}(w) = 0, \quad (4.86)$$

where $\tilde{F}(w)$ and $\tilde{G}(w)$ are defined earlier in (2.116). On putting $\tau = 0$ in (4.86) we get the corresponding secular equation (2.115) of TRDTE, as expected.

Low frequency expansions

This is similar to the previous sections concerning the low frequency limits. When $\omega \rightarrow 0$, the roots of the secular equation (4.86) are the zeros of $\tilde{F}(w) \equiv \tilde{p}_i$, where $i = 0, 1, 2, 3$, defining $\tilde{p}_0 \equiv 0$. Taylor expansions take the form

$$w_i(\omega) = \tilde{p}_i + \sum_{n=1}^{\infty} d_n^{(i)} (-i\omega)^n, \quad i = 0, 1, 2, 3. \quad (4.87)$$

When $i = 0, n = 1$

$$w_0(\omega) = \tilde{p}_0 + d_1^{(0)} (-i\omega) + O(\omega^2). \quad (4.88)$$

Substituting (4.88) into (4.86) we get

$$d_1^{(0)} = -\tilde{\sigma}^2 \frac{\tilde{W}_1 \tilde{W}_2}{\tilde{p}_1 \tilde{p}_2 \tilde{p}_3}. \quad (4.89)$$

The sign of $d_1^{(0)}$ depends on the sign of \tilde{p}_1 . If $\tilde{p}_1 > 0$ then $d_1^{(0)} < 0$, thus $w_0(\omega)$ is unstable. But stability is satisfied if $d_1^{(0)} > 0$ and this obtained if $\tilde{p}_1 < 0$. As before, $\tilde{p}_1 = 0$ is a special case.

Substituting (4.89) into (4.88) gives

$$w_0(\omega) = i\omega\tilde{\sigma}^2 \frac{\tilde{G}(0)}{\tilde{F}'(0)} + O(\omega^2). \quad (4.90)$$

When $i = 1, n = 1$ we obtain

$$w_1(\omega) = \tilde{p}_1 + d_1^{(1)}(-i\omega) + O(\omega^2). \quad (4.91)$$

Substituting (4.91) into (4.86) we get

$$d_1^{(1)} = -\tilde{\sigma} \left[\tilde{p}_1 \left(\varepsilon^{\frac{1}{2}} \alpha_1 + \tilde{\sigma}(\alpha_0 + \tau) \right) - \tilde{\sigma} \right] \frac{(\tilde{p}_1 - \tilde{W}_1)(\tilde{p}_1 - \tilde{W}_2)}{\tilde{p}_1(\tilde{p}_1 - \tilde{p}_2)(\tilde{p}_1 - \tilde{p}_3)}. \quad (4.92)$$

The sign of $d_1^{(1)}$ depends on the signs of $\left[\tilde{p}_1 \left(\varepsilon^{\frac{1}{2}} \alpha_1 + \tilde{\sigma}(\alpha_0 + \tau) \right) - \tilde{\sigma} \right]$ and \tilde{p}_1 . Stability is satisfied if $d_1^{(1)} > 0$, and $d_1^{(1)}$ is positive if

$$0 < \tilde{p}_1 < \frac{\tilde{\sigma}}{\varepsilon^{\frac{1}{2}} \alpha_1 + \tilde{\sigma}(\alpha_0 + \tau)}.$$

Inserting (4.92) into (4.91) gives

$$w_1(\omega) = \tilde{p}_1 + i\omega\tilde{\sigma} \left[\tilde{p}_1 \left(\varepsilon^{\frac{1}{2}} \alpha_1 + \tilde{\sigma}(\alpha_0 + \tau) \right) - \tilde{\sigma} \right] \frac{\tilde{G}(\tilde{p}_1)}{\tilde{F}'(\tilde{p}_1)} + O(\omega^2). \quad (4.93)$$

Similarly, when $i = 2, 3, n = 1$, we obtain

$$w_2(\omega) = \tilde{p}_1 + d_1^{(2)}(-i\omega) + O(\omega^2), \quad (4.94)$$

$$w_3(\omega) = \tilde{p}_1 + d_1^{(3)}(-i\omega) + O(\omega^2). \quad (4.95)$$

Substituting (4.94) and (4.95) into (4.86) respectively we get

$$d_1^{(2)} = -\tilde{\sigma} \left[\tilde{p}_2 \left(\varepsilon^{\frac{1}{2}} \alpha_1 + \tilde{\sigma}(\alpha_0 + \tau) \right) - \tilde{\sigma} \right] \frac{(\tilde{p}_2 - \tilde{W}_1)(\tilde{p}_2 - \tilde{W}_2)}{\tilde{p}_2(\tilde{p}_2 - \tilde{p}_1)(\tilde{p}_2 - \tilde{p}_3)}, \quad (4.96)$$

$$d_1^{(3)} = -\tilde{\sigma} \left[\tilde{p}_3 \left(\varepsilon^{\frac{1}{2}} \alpha_1 + \tilde{\sigma}(\alpha_0 + \tau) \right) - \tilde{\sigma} \right] \frac{(\tilde{p}_3 - \tilde{W}_1)(\tilde{p}_3 - \tilde{W}_2)}{\tilde{p}_3(\tilde{p}_3 - \tilde{p}_1)(\tilde{p}_3 - \tilde{p}_2)}. \quad (4.97)$$

The signs of $d_1^{(2)}$ and $d_1^{(3)}$ depend on the signs of

$$[\tilde{p}_2(\varepsilon^{\frac{1}{2}}\alpha_1 + \tilde{\sigma}(\alpha_0 + \tau)) - \tilde{\sigma}] \text{ and } [\tilde{p}_3(\varepsilon^{\frac{1}{2}}\alpha_1 + \tilde{\sigma}(\alpha_0 + \tau)) - \tilde{\sigma}],$$

respectively. The sign of \tilde{p}_1 does not matter here. Stability is satisfied if $d_1^{(2)}$ and $d_1^{(3)}$ are positive and this is obtained if

$$\tilde{p}_2 < \frac{\tilde{\sigma}}{\varepsilon^{\frac{1}{2}}\alpha_1 + \tilde{\sigma}(\alpha_0 + \tau)},$$

and

$$\tilde{p}_3 < \frac{\tilde{\sigma}}{\varepsilon^{\frac{1}{2}}\alpha_1 + \tilde{\sigma}(\alpha_0 + \tau)},$$

respectively. However, if $d_1^{(2)}$ and $d_1^{(3)}$ are negative, then $w_2(\omega)$ and $w_3(\omega)$ are unstable.

$$w_2(\omega) = \tilde{p}_2 + i\omega\tilde{\sigma} \left[\tilde{p}_1 \left(\varepsilon^{\frac{1}{2}}\alpha_1 + \tilde{\sigma}(\alpha_0 + \tau) \right) - \tilde{\sigma} \right] \frac{\tilde{G}(\tilde{p}_2)}{\tilde{F}'(\tilde{p}_2)} + O(\omega^2). \quad (4.98)$$

It is clear that, $w_2(\omega)$ is stable if,

$$\tilde{p}_2 < \frac{\tilde{\sigma}}{\varepsilon^{\frac{1}{2}}\alpha_1 + \tilde{\sigma}(\alpha_0 + \tau)}.$$

We have

$$w_3(\omega) = \tilde{p}_3 + i\omega\tilde{\sigma} \left[\tilde{p}_3 \left(\varepsilon^{\frac{1}{2}}\alpha_1 + \tilde{\sigma}(\alpha_0 + \tau) \right) - \tilde{\sigma} \right] \frac{\tilde{G}(\tilde{p}_3)}{\tilde{F}'(\tilde{p}_3)} + O(\omega^2). \quad (4.99)$$

Clearly, $w_3(\omega)$ is stable if,

$$\tilde{p}_3 < \frac{\tilde{\sigma}}{\varepsilon^{\frac{1}{2}}\alpha_1 + \tilde{\sigma}(\alpha_0 + \tau)}.$$

Summarising, branches $w_i(\omega)$, $i = 1, 2, 3$, are stable if the quantity

$$\tilde{p}_i[\varepsilon^{\frac{1}{2}}\alpha_1 + \tilde{\sigma}(\alpha_0 + \tau)] - \tilde{\sigma}$$

is negative. Conversely, these branches become unstable if this quantity is positive.

High frequency expansions

When $\omega \rightarrow \infty$, the roots of the secular equation in the high frequency limit are obtained by putting $\omega = \zeta^{-1}$ and taking $\zeta \rightarrow 0$. So, the secular equation may be written in terms of ζ as

$$\tilde{F}(w) - i\zeta^{-1}\tilde{\sigma} \left[w \left(\varepsilon^{\frac{1}{2}}\alpha_1 + \tilde{\sigma}(\alpha_0 + \tau(1 - i\zeta^{-1}\alpha_0)) \right) - \tilde{\sigma} \right] \tilde{G}(w) = 0. \quad (4.100)$$

Multiplying by ζ^2 we get

$$\zeta^2 \tilde{F}(w) - i\zeta \tilde{\sigma} \left[w \left(\varepsilon^{\frac{1}{2}} \alpha_1 + \tilde{\sigma}(\alpha_0 + \tau) - i\zeta^{-1} \alpha_0 \tilde{\sigma} \tau \right) - \tilde{\sigma} \right] \tilde{G}(w) = 0. \quad (4.101)$$

Rearranging this equation gives

$$\zeta^2 \tilde{F}(w) - \tilde{\sigma} \left[i\zeta \left(w \left(\varepsilon^{\frac{1}{2}} \alpha_1 + \tilde{\sigma}(\alpha_0 + \tau) \right) - \tilde{\sigma} \right) + w \tilde{\sigma} \alpha_0 \tau \right] \tilde{G}(w) = 0. \quad (4.102)$$

Putting $\zeta = 0$ we obtain

$$w \tilde{\sigma} \alpha_0 \tau \tilde{G}(w) = 0.$$

For $\tilde{\sigma} > 0$, $\alpha_0 > 0$ and $\tau > 0$ then the roots are

$$w = 0, \tilde{W}_1, \tilde{W}_2,$$

and the fourth one tends to infinity. Now look for the roots of the secular equation as $\zeta \rightarrow 0$, so the roots may be written as

$$w_1(\zeta) = A_3 \zeta, \quad (4.102a)$$

$$w_2(\zeta) = \tilde{W}_1 + B_3 \zeta, \quad (4.102b)$$

$$w_3(\zeta) = \tilde{W}_2 + D_2 \zeta, \quad (4.102c)$$

$$w_4(\zeta) = D_3 \zeta^{-1} + E_2 + F \zeta. \quad (4.102d)$$

We may get the roots of the secular equation in the high frequency limit, as $\zeta \rightarrow 0$, by finding the constants A_3, B_3, C_2, D_2, E_2 , by substituting into (4.102). Firstly, inserting (4.102a) into (4.102) we get

$$\begin{aligned} \zeta^2 (A_3 \zeta - \tilde{p}_1)(A_3 \zeta - \tilde{p}_2)(A_3 \zeta - \tilde{p}_3) - \tilde{\sigma} \left[i\zeta \left(A_3 \zeta \left(\varepsilon^{\frac{1}{2}} \alpha_1 + \tilde{\sigma}(\alpha_0 + \tau) \right) - \tilde{\sigma} \right) \right. \\ \left. + A_3 \zeta \tilde{\sigma} \alpha_0 \tau \right] (A_3 \zeta - \tilde{W}_1)(A_3 \zeta - \tilde{W}_2) = 0. \end{aligned}$$

The coefficients of ζ are

$$-\tilde{\sigma}^2(-i + A_3 \alpha_0 \tau) \tilde{W}_1 \tilde{W}_2 = 0.$$

Thus, for $\tilde{\sigma} > 0$, $\tilde{W}_1 > 0$ and $\tilde{W}_2 > 0$, find that

$$A_3 = \frac{i}{\alpha_0 \tau}.$$

Substituting A_3 into (4.102a) to get

$$w_1(\zeta) = \frac{i\zeta}{\alpha_0\tau} + O(\zeta^2).$$

It is clear that the stability condition is not satisfied because $\text{Im } w_1(\zeta) > 0$.

Inserting (4.102b) into (4.102) we get

$$\begin{aligned} & \zeta^2(\tilde{W}_1 + B_3\zeta)(\tilde{W}_1 - \tilde{p}_1 + B_3\zeta)(\tilde{W}_1 - \tilde{p}_2 + B_3\zeta)(\tilde{W}_1 - \tilde{p}_3 + B_3\zeta) \\ & - \tilde{\sigma} \left[i\zeta \left((\tilde{W}_1 + B_3\zeta)(\varepsilon^{\frac{1}{2}}\alpha_1 + \tilde{\sigma}(\alpha_0 + \tau)) - \tilde{\sigma} \right) + (\tilde{W}_1 + B_3\zeta)\tilde{\sigma}\alpha_0\tau \right] B_3\zeta(\tilde{W}_1 + B_3\zeta - \tilde{W}_2) = 0. \end{aligned}$$

Cancelling ζ , this equation becomes

$$\begin{aligned} & \zeta(\tilde{W}_1 + B_3\zeta)(\tilde{W}_1 - \tilde{p}_1 + B_3\zeta)(\tilde{W}_1 - \tilde{p}_2 + B_3\zeta)(\tilde{W}_1 - \tilde{p}_3 + B_3\zeta) \\ & - \tilde{\sigma} \left[i\zeta \left((\tilde{W}_1 + B_3\zeta)(\varepsilon^{\frac{1}{2}}\alpha_1 + \tilde{\sigma}(\alpha_0 + \tau)) - \tilde{\sigma} \right) + (\tilde{W}_1 + B_3\zeta)\tilde{\sigma}\alpha_0\tau \right] B_3(\tilde{W}_1 + B_3\zeta - \tilde{W}_2) = 0. \end{aligned}$$

Now putting $\zeta = 0$ we get

$$-i\tilde{\sigma} \left[-\tilde{W}_1\tilde{\sigma}\alpha_0i \right] B_3(\tilde{W}_1 - \tilde{W}_2) = 0.$$

We find that

$$B_3 = 0.$$

So we need to rewrite to (4.102b) as

$$w_2(\zeta) = \tilde{W}_1 + B_3\zeta + C_2\zeta^2,$$

and $B_3 = 0$, so w_2 becomes

$$w_2(\zeta) = \tilde{W}_1 + C_2\zeta^2. \tag{4.102e}$$

Now we want to find C_2 by substituting (4.102e) into (4.102) to get

$$\begin{aligned} & \zeta^2(\tilde{W}_1 + C_2\zeta^2)(\tilde{W}_1 - \tilde{p}_1 + C_2\zeta^2)(\tilde{W}_1 - \tilde{p}_2 + C_2\zeta^2)(\tilde{W}_1 - \tilde{p}_3 + C_2\zeta^2) \\ & - \tilde{\sigma} \left[i\zeta \left((\tilde{W}_1 + C_2\zeta^2)(\varepsilon^{\frac{1}{2}}\alpha_1 + \tilde{\sigma}(\alpha_0 + \tau)) - \tilde{\sigma} \right) + (\tilde{W}_1 + C_2\zeta^2)\tilde{\sigma}\alpha_0\tau \right] C_2\zeta^2(\tilde{W}_1 + C_2\zeta^2 - \tilde{W}_2) = 0. \end{aligned}$$

Cancel ζ^2 to get

$$\begin{aligned} & (\tilde{W}_1 + C_2\zeta^2)(\tilde{W}_1 - \tilde{p}_1 + C_2\zeta^2)(\tilde{W}_1 - \tilde{p}_2 + C_2\zeta^2)(\tilde{W}_1 - \tilde{p}_3 + C_2\zeta^2) \\ & - \tilde{\sigma} \left[i\zeta \left((\tilde{W}_1 + C_2\zeta^2)(\varepsilon^{\frac{1}{2}}\alpha_1 + \tilde{\sigma}(\alpha_0 + \tau)) - \tilde{\sigma} \right) + (\tilde{W}_1 + C_2\zeta^2)\tilde{\sigma}\alpha_0\tau \right] C_2(\tilde{W}_1 + C_2\zeta^2 - \tilde{W}_2) = 0. \end{aligned} \tag{4.102f}$$

Putting $\zeta = 0$ we obtain

$$C_2 = \frac{\tilde{W}_1(\tilde{W}_1 - \tilde{p}_1)(\tilde{W}_1 - \tilde{p}_2)(\tilde{W}_1 - \tilde{p}_3)}{\tilde{\sigma}^2 \tau \alpha_0 \tilde{W}_1(\tilde{W}_1 - \tilde{W}_2)}. \quad (4.102g)$$

It is clear that C_2 is real so we need to extend (4.102e) to include a term in ζ^3 :

$$w_2(\zeta) = \tilde{W}_1 + C_2 \zeta^2 + C_3 \zeta^3. \quad (4.102h)$$

Now we need to find C_3 by inserting (4.102h) into (4.102), and obtain

$$C_3 = \frac{-iC_2[\tilde{W}_1(\varepsilon^{1/2}\alpha_1 + \tilde{\sigma}(\alpha_0 + \tau)) - \tilde{\sigma}]}{\tilde{W}_1 \tilde{\sigma} \alpha_0 \tau}.$$

With the aid of (4.102g) the second root becomes

$$w_2(\zeta) = \tilde{W}_1 + \frac{\zeta^2 \tilde{F}(\tilde{W}_1)}{\tilde{\sigma}^2 \tau \alpha_0 \tilde{G}'(\tilde{W}_1)} \left\{ 1 - i\zeta \left(\frac{\tilde{W}_1[\varepsilon^{1/2}\alpha_1 + \tilde{\sigma}(\alpha_0 + \tau)] - \tilde{\sigma}}{\tilde{W}_1 \tilde{\sigma} \alpha_0 \tau} \right) \right\} + O(\zeta^4).$$

Similarly for w_3 , by inserting (4.102c) into (4.102) we obtain

$$D_2 = \frac{\tilde{W}_2(\tilde{W}_2 - \tilde{p}_1)(\tilde{W}_2 - \tilde{p}_2)(\tilde{W}_2 - \tilde{p}_3)}{\tilde{\sigma}^2 \tau \alpha_0 \tilde{W}_2(\tilde{W}_2 - \tilde{W}_1)}. \quad (4.102i)$$

It is clear that D_2 is real, so we need to rewrite (4.102c) as

$$w_3(\zeta) = \tilde{W}_2 + D_2 \zeta^2 + D_3 \zeta^3. \quad (4.102j)$$

Again, substituting (4.102j) into (4.102) we get

$$D_3 = \frac{-iD_2[\tilde{W}_2(\varepsilon^{1/2}\alpha_1 + \tilde{\sigma}(\alpha_0 + \tau)) - \tilde{\sigma}]}{\tilde{W}_2 \tilde{\sigma} \alpha_0 \tau}.$$

With the aid of (4.102i) the third root is written as

$$w_3(\zeta) = \tilde{W}_2 + \frac{\zeta^2 \tilde{F}(\tilde{W}_2)}{\tilde{\sigma}^2 \tau \alpha_0 \tilde{G}'(\tilde{W}_2)} \left\{ 1 - i\zeta \left(\frac{\tilde{W}_2[\varepsilon^{1/2}\alpha_1 + \tilde{\sigma}(\alpha_0 + \tau)] - \tilde{\sigma}}{\tilde{W}_2 \tilde{\sigma} \alpha_0 \tau} \right) \right\} + O(\zeta^4).$$

Finally, the fourth root may obtained by substituting (4.102d) into (4.102) to get

$$\begin{aligned} & \zeta^2(D_3\zeta^{-1} + E_2)(D_3\zeta^{-1} + E_2 - \tilde{p}_1)(D_3\zeta^{-1} + E_2 - \tilde{p}_2)(D_3\zeta^{-1} + E_2 - \tilde{p}_3) \\ & - \tilde{\sigma} \left[i\zeta \left((D_3\zeta^{-1} + E_2)(\varepsilon^{\frac{1}{2}}\alpha_1 + \tilde{\sigma}(\alpha_0 + \tau)) - \tilde{\sigma} \right) + (D_3\zeta^{-1} + E_2)\tilde{\sigma}\alpha_0\tau \right] \\ & (D_3\zeta^{-1} + E_2 - \tilde{W}_1)(D_3\zeta^{-1} + E_2 - \tilde{W}_2) = 0. \end{aligned}$$

Multiplying by ζ^3 we obtain

$$\begin{aligned} & \zeta(D_3 + E_2\zeta)[D_3 + (E_2 - \tilde{p}_1)\zeta][D_3 + (E_2 - \tilde{p}_2)\zeta][D_3 + (E_2 - \tilde{p}_3)\zeta] \\ & - \tilde{\sigma} \left[i \left((D_3\zeta + E_2\zeta^2)(\varepsilon^{\frac{1}{2}}\alpha_1 + \tilde{\sigma}(\alpha_0 + \tau)) - \tilde{\sigma}\zeta^2 \right) + (D_3 + E_2\zeta)\tilde{\sigma}\alpha_0\tau \right] \\ & [D_3 + (E_2 - \tilde{W}_1)\zeta][D_3 + (E_2 - \tilde{W}_2)\zeta] = 0. \end{aligned}$$

Putting $\zeta = 0$ we get

$$-\tilde{\sigma}^2\alpha_0\tau D_3^3 = 0.$$

Since $\tilde{\sigma} > 0, \alpha_0 > 0$ and for $\tau > 0$ we find that

$$D_3 = 0.$$

So, now we need to find E_2 and F in equation (4.102d). Substituting

$$w_4 = E_2 + F\zeta$$

into (4.102) we get

$$\begin{aligned} & \zeta^2(E_2 + F\zeta)(E_2 + F\zeta - \tilde{p}_1)(E_2 + F\zeta - \tilde{p}_2)(E_2 + F\zeta - \tilde{p}_3) \\ & - \tilde{\sigma} \left[i\zeta \left((E_2 + F\zeta)(\varepsilon^{\frac{1}{2}}\alpha_1 + \tilde{\sigma}(\alpha_0 + \tau)) - \tilde{\sigma} \right) + (E_2 + F\zeta)\tilde{\sigma}\alpha_0\tau \right] \\ & (E_2 + F\zeta - \tilde{W}_1)(E_2 + F\zeta - \tilde{W}_2) = 0. \end{aligned}$$

The constants terms must vanish to find E_2 , thus

$$-\tilde{\sigma}^2 E_2 \alpha_0 \tau (E_2 - \tilde{W}_1)(E_2 - \tilde{W}_2) = 0,$$

so

$$E_2 = 0, \tilde{W}_1, \tilde{W}_2.$$

The last two values correspond to the first two branches we already found. If we substitute $E_2 = 0$ into the secular equation (4.102) to get F we will find that $F = A_3$ similar to the first branch. However, there must be a root tending to infinity as $\omega \rightarrow \infty$, i.e. $\zeta \rightarrow 0$. Therefore, suppose that the fourth root may be written as

$$w_4 = D_3\zeta^{-n}, \quad \text{some } n > 0. \quad (4.102k)$$

Substituting (4.102k) into (4.102) gives

$$D_3^4 \zeta^{2-4n} - D_3^2 \left[i \left(\tilde{\sigma}(\varepsilon^{1/2} \alpha_1 + \tilde{\sigma}(\alpha_0 + \tau)) D_3 \zeta^{1-3n} - \tilde{\sigma}^2 \zeta^{1-2n} \right) + D_3 \tilde{\sigma}^2 \alpha_0 \tau \zeta^{-3n} \right] = 0.$$

Multiplying by ζ^{4n} :

$$D_3^4 \zeta^2 - D_3^2 \left[i \left(\tilde{\sigma}(\varepsilon^{1/2} \alpha_1 + \tilde{\sigma}(\alpha_0 + \tau)) D_3 \zeta^{1+n} - \tilde{\sigma}^2 \zeta^{1+2n} \right) + D_3 \tilde{\sigma}^2 \alpha_0 \tau \zeta^n \right] = 0. \quad (4.102l)$$

If we take $n = 1$ we get $D_3 = 0$, exactly as before. The only meaningful balance of terms seems to be between the first and last terms:

$$D_3^4 \zeta^2 - D_3^3 \tilde{\sigma}^2 \alpha_0 \tau \zeta^n = 0. \quad (4.102m)$$

So we must take $n = 2$, leading to roots

$$D_3 = 0, 0, 0, \tilde{\sigma}^2 \alpha_0 \tau.$$

With $n = 2$, (4.102l) becomes

$$D_3^4 \zeta^2 - D_3^2 \left[i \left(\tilde{\sigma}(\varepsilon^{1/2} \alpha_1 + \tilde{\sigma}(\alpha_0 + \tau)) D_3 \zeta^3 - \tilde{\sigma}^2 \zeta^5 \right) + D_3 \tilde{\sigma}^2 \alpha_0 \tau \zeta^2 \right] = 0.$$

We can ignore the ζ^3 and ζ^5 terms as they are smaller than ζ^2 as $\zeta \rightarrow 0$. This then gives (4.102m) with $n = 2$. There might now be a non-zero term proportional to ζ^{-1} , so to get this term we need to rewrite the fourth root as

$$w_4 = D_3 \zeta^{-2} + D_4 \zeta^{-1}, \quad \text{where } D_3 = \tilde{\sigma}^2 \alpha_0 \tau. \quad (4.102n)$$

Insert (4.102n) into (4.102) to get

$$D_4 = i \tilde{\sigma} \left[\varepsilon^{1/2} \alpha_1 + \tilde{\sigma}(\alpha_0 + \tau) \right].$$

Now the fourth branch becomes

$$w_4 = \tilde{\sigma}^2 \alpha_0 \tau \zeta^{-2} + i \tilde{\sigma} \left[\varepsilon^{1/2} \alpha_1 + \tilde{\sigma}(\alpha_0 + \tau) \right] \zeta^{-1} + O(1),$$

with $\text{Im } w_4 > 0$, corresponding to instability.

Numerical results

In Figure 4.3 we have taken $\tilde{p}_1 > 0$. The branch $w_0(\omega)$ beginning at the origin is

unstable in the low and high frequencies in each part of the Figure. All the other branches begin to the right of this branch. If α_0 and α_1 are small enough then

$$\tilde{p}_i[\varepsilon^{\frac{1}{2}}\alpha_1 + \tilde{\sigma}(\alpha_0 + \tau)] - \tilde{\sigma} < 0, \quad \text{for } i = 1, 2, 3,$$

and

$$\tilde{W}_i[\varepsilon^{1/2}\alpha_1 + \tilde{\sigma}(\alpha_0 + \tau)] - \tilde{\sigma} > 0, \quad \text{for } i = 1, 2.$$

and so all the branches $w_i(\omega)$, $i = 1, 2, 3$, are stable in the low and high frequencies except $w_3(\omega)$ is unstable in the high frequency. This can be seen in the first subfigure (a) of Figure 4.3 where α_0 and α_1 are small. As α_0 and α_1 increase, first $w_2(\omega)$ becomes unstable, see part (c), and as they increase further $w_1(\omega)$ becomes unstable.

In Figure 4.4 we have taken $\tilde{p}_1 < 0$. The branch $w_1(\omega)$ beginning at $w = \tilde{p}_1$ is unstable in each part of the Figure. All the other branches begin to the right of this branch. The branch $w_0(\omega)$ begins at the origin and is stable in each part of the Figure in the low frequency but is unstable in the high frequency with increasing of α_0 and α_1 . As in Figure 4.3, increasing α_0 and α_1 leads to increasing instability for other branches.

In Figure 4.5 we illustrate the exceptional case $\tilde{p}_1 = 0$. Now two branches emanate from the origin, namely, $w_0(\omega)$ and $w_1(\omega)$, one unstable and the other stable in the low frequency but in the high frequency $w_0(\omega)$ maintains the instability and $w_1(\omega)$ becomes also unstable, one with argument $-\pi/4$ and the other with argument $3\pi/4$. The same increasing instability with increasing α_0 and α_1 is observed.

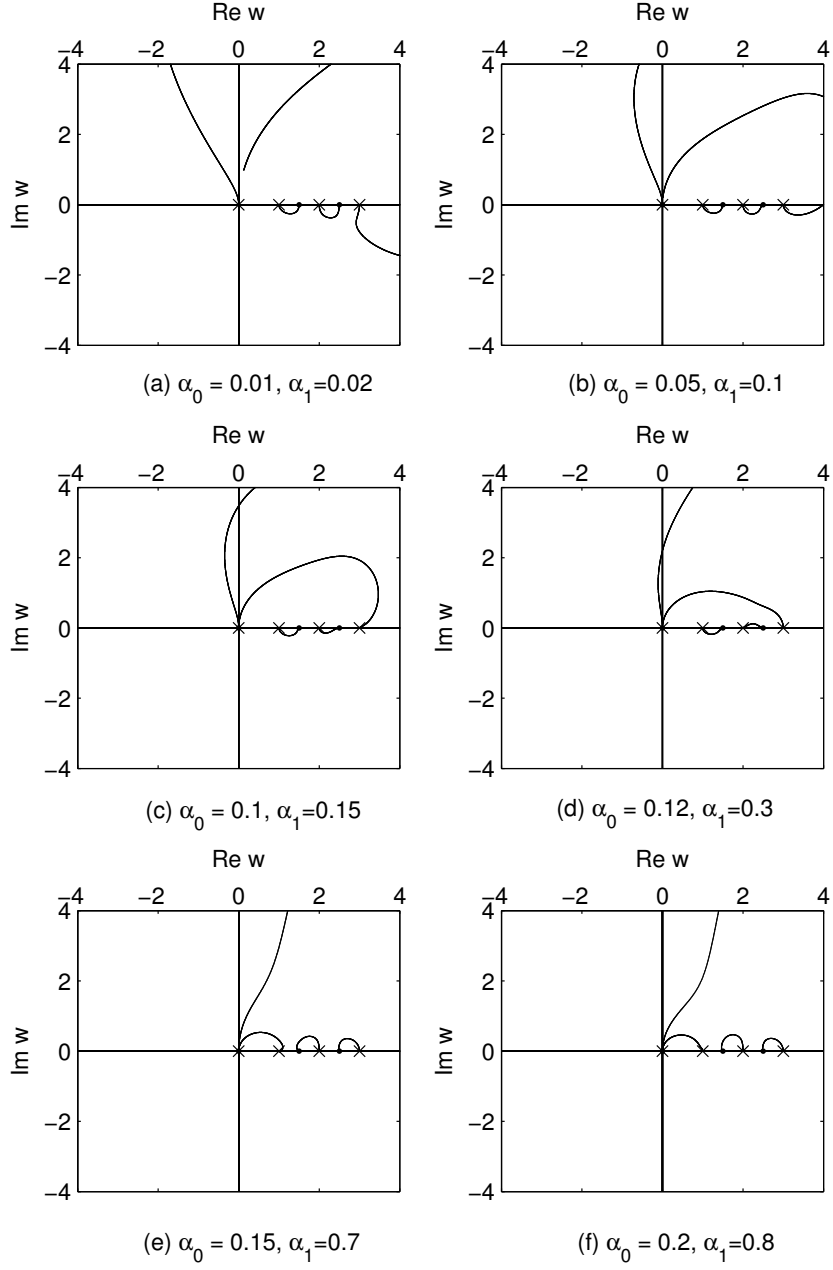


Figure 4.3: The longitudinal squared wave speeds of constrained anisotropic TRDTE+GTE(2) theory. For each part, $\tilde{p}_1 = 1, \tilde{p}_2 = 2, \tilde{p}_3 = 3, \tilde{W}_1 = 1.5, \tilde{W}_2 = 2.5, \tilde{\sigma} = 1, \varepsilon = 1$.

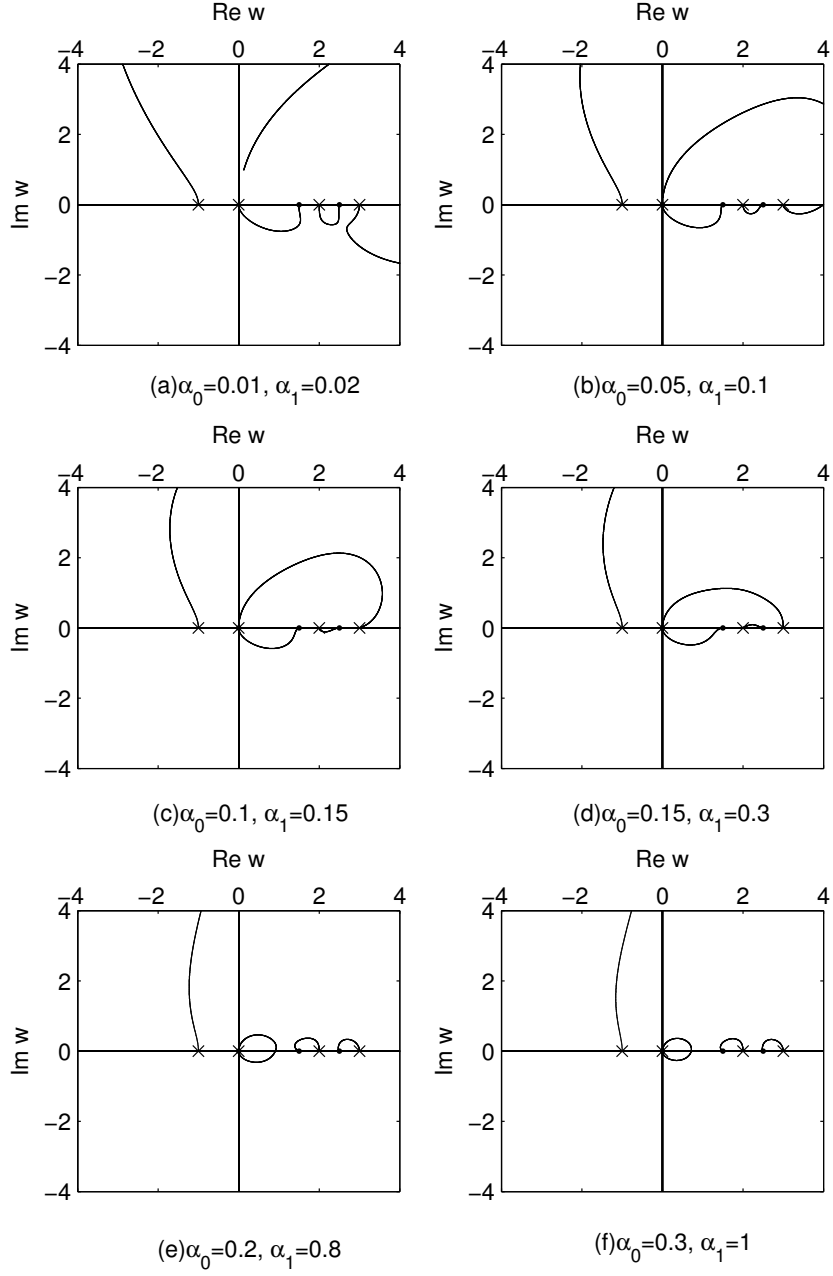


Figure 4.4: The longitudinal squared wave speeds of constrained anisotropic TRDTE+GTE(2) theory. For each part, $\tilde{p}_1 = -1$, $\tilde{p}_2 = 2$, $\tilde{p}_3 = 3$, $\tilde{W}_1 = 1.5$, $\tilde{W}_2 = 2.5$, $\tilde{\sigma} = 1$, $\varepsilon = 1$.

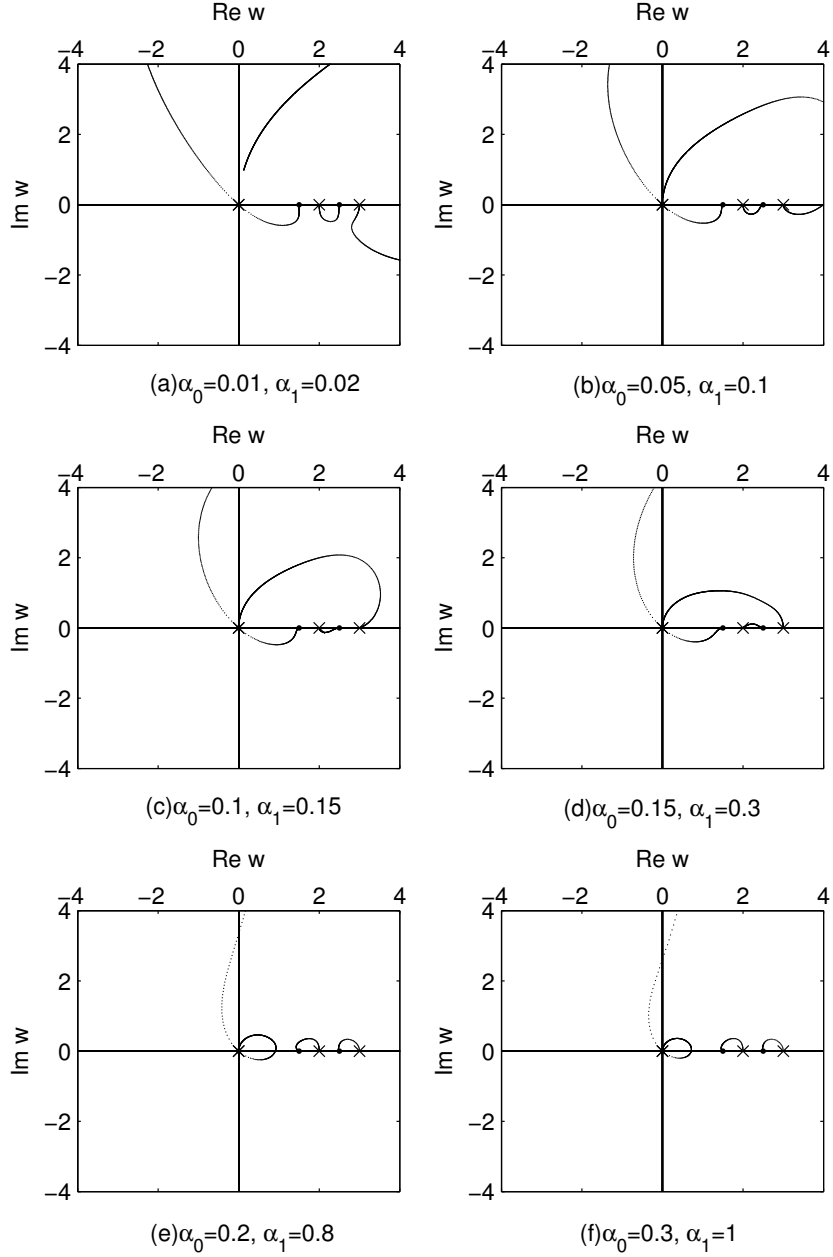


Figure 4.5: The longitudinal squared wave speeds of constrained anisotropic TRDTE+GTE(2) theory. For each part, $\tilde{p}_1 = 0$, $\tilde{p}_2 = 2$, $\tilde{p}_3 = 3$, $\tilde{W}_1 = 1.5$, $\tilde{W}_2 = 2.5$, $\tilde{\sigma} = 1$, $\varepsilon = 1$.

4.3.2 Alternative form of deformation temperature constraint

In this section we will use equations (4.67), (4.70) and (2.144),

$$\left. \begin{aligned} \tilde{c}_{ijkl}u_{k,jl} - \beta_{ij}(\theta_{,j} + \alpha_1\dot{\theta}_{,j}) + \tilde{N}_{ij}\tilde{\eta}_{,j} &= \rho\ddot{u}_i, \\ k_{pq}\theta_{,pq} - T\beta_{pq}\left(1 + \tau\frac{\partial}{\partial t}\right)\dot{u}_{p,q} - \rho c(\dot{\theta} + (\alpha_0 + \tau)\ddot{\theta} + \alpha_0\tau\ddot{\ddot{\theta}}) - T\alpha(\dot{\tilde{\eta}} + \tau\ddot{\tilde{\eta}}) &= 0, \\ \tilde{N}_{pq}u_{p,q} - \alpha(\theta + \alpha_0\dot{\theta}) &= 0. \end{aligned} \right\} \quad (4.103)$$

Now we follow the same steps of Section 2.3.2 and 3.3.2 to get the secular equation. Firstly, looking for solution for equations (4.103) in the form of the plane harmonic waves (2.86) by inserting (2.86) into (4.103) and we will get the same equations we had before (4.73), (4.74) and (2.146)₃

$$\left. \begin{aligned} (\tilde{Q}_{ik} - \rho s^{-2}\delta_{ik})U_k + i(\omega s)^{-1}[b_i(1 - i\omega\alpha_1)\Theta - \tilde{c}_i\tilde{H}] &= 0, \\ \omega s T b_p(1 - i\omega\tau)U_p - i\alpha T(1 - i\omega\tau)\tilde{H} + (\omega s^2k - i\rho c(1 - i(\alpha_0 + \tau)\omega - \alpha_0\tau\omega^2))\Theta &= 0, \\ i\omega s\tilde{c}_pU_p - \alpha(1 - i\omega\alpha_0)\Theta &= 0, \end{aligned} \right\} \quad (4.104)$$

where $\tilde{N}_{pq}n_q = \tilde{c}_p$, $\beta_{ij}n_j = b_i$, $k_{pq}n_pn_q = k$. Eliminate Θ and \tilde{H} between (4.104). From (4.104)₃ we find that

$$\Theta = \frac{i\omega s\tilde{c}_pU_p}{\alpha(1 - i\omega\alpha_0)}. \quad (4.105)$$

Substituting (4.105) into (4.104)₂ we get

$$\tilde{H} = -i\alpha^{-1}\omega s b_pU_p + \frac{\omega s\tilde{c}_pU_p}{(\alpha^2T)(1 - i\omega\alpha_0)(1 - i\omega\tau)}\left(\omega s^2k - i\rho c(1 - i(\alpha_0 + \tau)\omega - \alpha_0\tau\omega^2)\right). \quad (4.106)$$

Inserting (4.105) and (4.106) into (4.104)₁ we obtain

$$\left\{ (\tilde{Q}_{ip} - \rho s^{-2}\delta_{ip}) - \alpha^{-1}(1 - i\omega\alpha_0)^{-1}(b_i\tilde{c}_p(1 - i\omega\alpha_1) + \tilde{c}_ib_p(1 - i\omega\alpha_0)) \right. \\ \left. - i(\alpha^2T)^{-1}(1 - i\omega\alpha_0)^{-1}(1 - i\omega\tau)^{-1}(\omega s^2k - i\rho c(1 - i(\alpha_0 + \tau)\omega - \alpha_0\tau\omega^2))\tilde{c}_i\tilde{c}_p \right\} U_p = 0. \quad (4.107)$$

Expanding this equation we get

$$\left\{ \tilde{Q}_{ip} - \alpha^{-1}(1 - i\omega\alpha_0)^{-1}\left[(b_i\tilde{c}_p + \tilde{c}_ib_p) - i\omega(\alpha_1b_i\tilde{c}_p + \alpha_0\tilde{c}_ib_p)\right] \right. \\ \left. - i(\alpha^2T)^{-1}(1 - i\omega\alpha_0)^{-1}(1 - i\omega\tau)^{-1}(\omega s^2k - i\rho c - \rho c(\alpha_0 + \tau)\omega + i\alpha_0\omega^2\tau\rho c)\tilde{c}_i\tilde{c}_p - \rho s^{-2}\delta_{ip} \right\} U_p = 0. \quad (4.108)$$

Rearranging this equation we get

$$\left\{ \tilde{Q}_{ip} - \alpha^{-1}(1 - i\omega\alpha_0)^{-1}(b_i\tilde{c}_p + \tilde{c}_i b_p) - (\alpha^2 T)^{-1}(1 - i\omega\alpha_0)^{-1}(1 - i\omega\tau)^{-1}\rho c\tilde{c}_i\tilde{c}_p + i\omega(1 - i\omega\alpha_0)^{-1} \right. \\ \left. \left[\alpha^{-1}(\alpha_1 b_i\tilde{c}_p + \alpha_0\tilde{c}_i b_p - (\alpha^2 T)^{-1}(1 - i\omega\tau)^{-1}(s^2 k - \rho c(\alpha_0 + \tau) + i\omega\tau\alpha_0\rho c)\tilde{c}_p\tilde{c}_i \right] - w\delta_{ip} \right\} U_p = 0.$$

The non-zero amplitudes satisfy (4.104) if and only if

$$\det \left\{ \tilde{\mathbf{Q}} - \alpha^{-1}(1 - i\omega\alpha_0)^{-1}(\mathbf{b} \otimes \tilde{\mathbf{c}} + \tilde{\mathbf{c}} \otimes \mathbf{b}) - (\alpha^2 T)^{-1}(1 - i\omega\alpha_0)^{-1}(1 - i\omega\tau)^{-1}\rho c\tilde{\mathbf{c}} \otimes \tilde{\mathbf{c}} + \right. \\ \left. i\omega(1 - i\omega\alpha_0)^{-1} \left[\alpha^{-1}(\alpha_1 \mathbf{b} \otimes \tilde{\mathbf{c}} + \alpha_0 \tilde{\mathbf{c}} \otimes \mathbf{b} - (\alpha^2 T)^{-1}(1 - i\omega\tau)^{-1}(s^2 k - \rho c(\alpha_0 + \tau) + i\omega\alpha_0\tau\rho c)\tilde{\mathbf{c}} \otimes \tilde{\mathbf{c}} \right] \right. \\ \left. - w\mathbf{1} \right\} = 0. \quad (4.109)$$

By defining

$$\tilde{\mathbf{S}}_1 := \tilde{\mathbf{Q}} - (1 - i\omega\alpha_0)^{-1} \left[\alpha^{-1}(\mathbf{b} \otimes \tilde{\mathbf{c}} + \tilde{\mathbf{c}} \otimes \mathbf{b}) + \frac{\rho c}{\alpha^2 T(1 - i\omega\tau)} \tilde{\mathbf{c}} \otimes \tilde{\mathbf{c}} \right] \quad (4.110)$$

equation (4.109) may be written as

$$\det \left\{ (\tilde{\mathbf{S}}_1 - w\mathbf{1}) + \frac{i\omega}{(1 - i\omega\alpha_0)} \left[\frac{\alpha_1}{\alpha} \mathbf{b} \otimes \tilde{\mathbf{c}} + \frac{\alpha_0}{\alpha} \tilde{\mathbf{c}} \otimes \mathbf{b} - \frac{(s^2 k - \rho c(\alpha_0 + \tau) + i\omega\alpha_0\tau\rho c)}{\alpha^2 T(1 - i\omega\tau)} \tilde{\mathbf{c}} \otimes \tilde{\mathbf{c}} \right] \right\} = 0. \quad (4.111)$$

In terms of definitions (2.110a) and (2.110b) $\tilde{\mathbf{S}}_1$ may be written as

$$\tilde{\mathbf{S}}_1 := \tilde{\mathbf{Q}} - (1 - i\omega\alpha_0)^{-1} \left[2\alpha^{-1}\beta + \frac{\rho c}{\alpha^2 T(1 - i\omega\tau)} \right] \mathbf{n} \otimes \mathbf{n}. \quad (4.112)$$

In dimensionless form $\tilde{\mathbf{S}}_1$ becomes

$$\tilde{\mathbf{S}}_1 := \tilde{\mathbf{Q}} - (1 - i\omega\alpha_0)^{-1} \tilde{\sigma} \left[2\varepsilon^{\frac{1}{2}} + \tilde{\sigma}(1 - i\omega\tau)^{-1} \right] \mathbf{n} \otimes \mathbf{n}. \quad (4.113)$$

Rewrite this definition as

$$\tilde{\mathbf{S}}_1 := \tilde{\mathbf{Q}} - (1 - i\omega\alpha_0)^{-1}(1 - i\omega\tau)^{-1} \tilde{\sigma} \left[2\varepsilon^{\frac{1}{2}}(1 - i\omega\tau) + \tilde{\sigma} \right] \mathbf{n} \otimes \mathbf{n}. \quad (4.114)$$

The secular equation (4.111) can be written in terms of definitions (2.110a) and (2.110b) as

$$\det \left\{ (\tilde{\mathbf{S}}_1 - w\mathbf{1}) + \frac{i\omega}{(1 - i\omega\alpha_0)} \left[\alpha^{-1}\beta(\alpha_1 + \alpha_0) - \frac{(s^2 k - \rho c(\alpha_0 + \tau) + i\omega\alpha_0\tau\rho c)}{\alpha^2 T(1 - i\omega\tau)} \right] \mathbf{n} \otimes \mathbf{n} \right\} = 0. \quad (4.115)$$

By using the standard identity (2.60), the secular equation becomes

$$\det(\tilde{\mathbf{S}}_1 - w\mathbf{1}) + \frac{i\omega}{(1 - i\omega\alpha_0)} \left[\alpha^{-1}\beta(\alpha_1 + \alpha_0) - \frac{(s^2k - \rho c(\alpha_0 + \tau) + i\omega\alpha_0\tau\rho c)}{\alpha^2 T(1 - i\omega\tau)} \right] \\ \mathbf{n} \cdot (\tilde{\mathbf{S}}_1 - w\mathbf{1})^{\text{adj}} \mathbf{n} = 0. \quad (4.116)$$

The non-dimensional form of the secular equation (4.116) is

$$w \det(w\mathbf{1} - \tilde{\mathbf{S}}_1) - \frac{i\omega\tilde{\sigma}}{(1 - i\omega\alpha_0)} \left[w\varepsilon^{\frac{1}{2}}(\alpha_1 + \alpha_0) - \tilde{\sigma}(1 - i\omega\tau)^{-1} \left(1 - w(\alpha_0 + \tau(1 - i\omega\alpha_0)) \right) \right] \\ \mathbf{n} \cdot (w\mathbf{1} - \tilde{\mathbf{S}}_1)^{\text{adj}} \mathbf{n} = 0. \quad (4.117)$$

Now the first term of (4.117) may be written as

$$M_1 \equiv w \det(w\mathbf{1} - \tilde{\mathbf{S}}_1) = w \det \left\{ w\mathbf{1} - \tilde{\mathbf{Q}} + \tilde{\sigma}(2\varepsilon^{\frac{1}{2}} + \tilde{\sigma})\mathbf{n} \otimes \mathbf{n} \right. \\ \left. + \left(-1 + \frac{1}{(1 - i\omega\alpha_0)(1 - i\omega\tau)} \right) \tilde{\sigma}(2\varepsilon^{\frac{1}{2}} + \tilde{\sigma})\mathbf{n} \otimes \mathbf{n} - \frac{2i\omega\tilde{\sigma}\tau\varepsilon^{\frac{1}{2}}}{(1 - i\omega\alpha_0)(1 - i\omega\tau)} \mathbf{n} \otimes \mathbf{n} \right\}.$$

After simplifying we get

$$M_1 = w \det \left\{ (w\mathbf{1} - \tilde{\mathbf{P}}) + \frac{i\omega\tilde{\sigma}}{(1 - i\omega\alpha_0)(1 - i\omega\tau)} [\alpha_0(1 - i\omega\tau)(2\varepsilon^{\frac{1}{2}} + \tilde{\sigma}) + \tilde{\sigma}\tau] \mathbf{n} \otimes \mathbf{n} \right\}.$$

Using the standard identity (2.60) we obtain

$$M_1 = w \det(w\mathbf{1} - \tilde{\mathbf{P}}) + \frac{i\omega\tilde{\sigma}w}{(1 - i\omega\alpha_0)(1 - i\omega\tau)} [\alpha_0(1 - i\omega\tau)(2\varepsilon^{\frac{1}{2}} + \tilde{\sigma}) + \tilde{\sigma}\tau] \mathbf{n} \cdot (w\mathbf{1} - \tilde{\mathbf{P}})^{\text{adj}} \mathbf{n}.$$

The last term of (4.117) may be written as

$$M_2 \equiv \mathbf{n} \cdot (w\mathbf{1} - \tilde{\mathbf{S}}_1)^{\text{adj}} \mathbf{n} = \mathbf{n} \cdot \left((w\mathbf{1} - \tilde{\mathbf{P}}) + \frac{i\omega\tilde{\sigma}}{(1 - i\omega\alpha_0)(1 - i\omega\tau)} \right. \\ \left. [\alpha_0(1 - i\omega\tau)(2\varepsilon^{\frac{1}{2}} + \tilde{\sigma}) + \tilde{\sigma}\tau] \mathbf{n} \otimes \mathbf{n} \right)^{\text{adj}} \mathbf{n}.$$

By using [28, (A2)] we get

$$M_2 = \mathbf{n} \cdot (w\mathbf{1} - \tilde{\mathbf{P}})^{\text{adj}} \mathbf{n}.$$

By substituting M_1 and M_2 into the secular equation (4.116) we get

$$w \det(w\mathbf{1} - \tilde{\mathbf{P}}) + \frac{i\omega\tilde{\sigma}w}{(1 - i\omega\alpha_0)(1 - i\omega\tau)} [\alpha_0(1 - i\omega\tau)(2\varepsilon^{\frac{1}{2}} + \tilde{\sigma}) + \tilde{\sigma}\tau] \mathbf{n} \cdot (w\mathbf{1} - \tilde{\mathbf{P}})^{\text{adj}} \mathbf{n} \\ - \frac{i\omega\tilde{\sigma}}{(1 - i\omega\alpha_0)} \left[w\varepsilon^{\frac{1}{2}}(\alpha_1 + \alpha_0) - \tilde{\sigma}(1 - i\omega\tau)^{-1} \left(1 - w(\alpha_0 + \tau(1 - i\omega\alpha_0)) \right) \right] \mathbf{n} \cdot (w\mathbf{1} - \tilde{\mathbf{P}})^{\text{adj}} \mathbf{n} = 0.$$

By simplifying and rearranging we obtain

$$w \det(w\mathbf{1} - \tilde{\mathbf{P}}) - \frac{i\omega\tilde{\sigma}}{(1 - i\omega\alpha_0)(1 - i\omega\tau)} [w\varepsilon^{\frac{1}{2}}(1 - i\omega\tau)(\alpha_1 - \alpha_0) - \tilde{\sigma}] \mathbf{n} \cdot (w\mathbf{1} - \tilde{\mathbf{P}})^{\text{adj}} \mathbf{n} = 0. \quad (4.118)$$

This is the secular equation for anisotropic TRDTE+GTE (2) which is constrained by the alternative deformation temperature constraint and has not previously appeared in the literature.

Now we need to rewrite equation (4.118) in terms of the eigenvalues $\tilde{p}_i, i = 1, 2, 3$, as

$$w(w - \tilde{p}_1)(w - \tilde{p}_2)(w - \tilde{p}_3) - \frac{i\omega\tilde{\sigma}}{(1 - i\omega\alpha_0)(1 - i\omega\tau)} [w\varepsilon^{\frac{1}{2}}(1 - i\omega\tau)(\alpha_1 - \alpha_0) - \tilde{\sigma}] \{n_1^2(w - \tilde{p}_2)(w - \tilde{p}_3) + n_2^2(w - \tilde{p}_1)(w - \tilde{p}_3) + n_3^2(w - \tilde{p}_1)(w - \tilde{p}_2)\} = 0. \quad (4.119)$$

Similarly to (3.140) the quadratic part in w within braces has zeros at $w = \tilde{W}_1, \tilde{W}_2$, which satisfy

$$\tilde{p}_1 < \tilde{W}_1 < \tilde{p}_2 < \tilde{W}_2 < \tilde{p}_3. \quad (4.118a)$$

Equation (4.119) may written as

$$\tilde{F}(w) - \frac{i\omega\tilde{\sigma}}{(1 - i\omega\alpha_0)(1 - i\omega\tau)} [w\varepsilon^{\frac{1}{2}}(1 - i\omega\tau)(\alpha_1 - \alpha_0) - \tilde{\sigma}] \tilde{G}(w) = 0, \quad (4.120)$$

where $\tilde{F}(w)$ and $\tilde{G}(w)$ are defined earlier in (2.28). Putting $\tau = 0$ we get the similar equation (2.168) in the TRDTE case.

Low frequency expansions

This is similar to the previous sections for the low-frequency limits. When $\omega \rightarrow 0$ the roots of the secular equation (4.120) are the zeros of $\tilde{F}(w) \equiv \tilde{p}_i, i = 0, 1, 2, 3$, defining $\tilde{p}_0 \equiv 0$. Taylor expansions take the form

$$w_i(\omega) = \tilde{p}_i + \sum_{n=1}^{\infty} d_n^{(i)} (-i\omega)^n, \quad i = 0, 1, 2, 3. \quad (4.121)$$

When $i = 0, n = 1$, the above equation may be put into the form

$$w_0(\omega) = \tilde{p}_0 + d_1^{(0)} (-i\omega) + O(\omega^2). \quad (4.122)$$

Substituting (4.122) into (4.120) we get

$$d_1^{(0)} = -\tilde{\sigma}^2 \frac{\tilde{W}_1 \tilde{W}_2}{\tilde{p}_1 \tilde{p}_2 \tilde{p}_3}. \quad (4.123)$$

The sign of $d_1^{(0)}$ depends on \tilde{p}_1 . Stability is satisfied if $d_1^{(0)} > 0$, and $d_1^{(0)}$ is positive if \tilde{p}_1 is negative. But if $\tilde{p}_1 > 0$ then $w_0(\omega)$ is unstable.

Inserting (4.123) into (4.122) to get

$$w_0(\omega) = i\omega\tilde{\sigma}^2 \frac{\tilde{G}(0)}{\tilde{F}'(0)} + O(\omega^2). \quad (4.124)$$

It is clear that $\text{Im } w_0(\omega) > 0$, thus $w_0(\omega)$ is unstable in the low frequency region.

When $i = 1, n = 1$ we obtain

$$w_1(\omega) = \tilde{p}_1 + d_1^{(1)}(-i\omega) + O(\omega^2). \quad (4.125)$$

Substituting (4.125) into (4.120) we get

$$d_1^{(1)} = -\tilde{\sigma} [\tilde{p}_1 \varepsilon^{\frac{1}{2}} (\alpha_1 - \alpha_0) - \tilde{\sigma}] \frac{(\tilde{p}_1 - \tilde{W}_1)(\tilde{p}_1 - \tilde{W}_2)}{\tilde{p}_1(\tilde{p}_1 - \tilde{p}_2)(\tilde{p}_1 - \tilde{p}_3)}. \quad (4.126)$$

The sign of $d_1^{(1)}$ depends on $[\tilde{p}_1 (\varepsilon^{\frac{1}{2}} (\alpha_1 + \alpha_0) + \tilde{\sigma}(\alpha_0 + \tau)) - \tilde{\sigma}]$ and \tilde{p}_1 . The stability condition is satisfied if $d_1^{(1)} > 0$ and $d_1^{(1)}$ is positive if

$$0 < \tilde{p}_1 < \frac{\tilde{\sigma}}{\varepsilon^{\frac{1}{2}} (\alpha_1 - \alpha_0)}.$$

Inserting (4.126) into (4.125) we get

$$w_1(\omega) = \tilde{p}_1 + i\omega\tilde{\sigma} \left[\tilde{p}_1 \varepsilon^{\frac{1}{2}} (\alpha_1 - \alpha_0) - \tilde{\sigma} \right] \frac{\tilde{G}(\tilde{p}_1)}{\tilde{F}'(\tilde{p}_1)} + O(\omega^2). \quad (4.127)$$

Similarly, when $i = 2, 3$

$$w_2(\omega) = \tilde{p}_2 + i\omega\tilde{\sigma} \left[\tilde{p}_2 \varepsilon^{\frac{1}{2}} (\alpha_1 - \alpha_0) - \tilde{\sigma} \right] \frac{\tilde{G}(\tilde{p}_2)}{\tilde{F}'(\tilde{p}_2)} + O(\omega^2). \quad (4.128)$$

It is clear that w_2 is stable if

$$\tilde{p}_2 < \frac{\tilde{\sigma}}{\varepsilon^{\frac{1}{2}} (\alpha_1 - \alpha_0)}.$$

We have similarly

$$w_3(\omega) = \tilde{p}_3 + i\omega\tilde{\sigma} \left[\tilde{p}_3 \varepsilon^{\frac{1}{2}} (\alpha_1 - \alpha_0) - \tilde{\sigma} \right] \frac{\tilde{G}(\tilde{p}_3)}{\tilde{F}'(\tilde{p}_3)} + O(\omega^2). \quad (4.129)$$

Also, w_3 is stable if

$$\tilde{p}_3 < \frac{\tilde{\sigma}}{\varepsilon^{\frac{1}{2}} (\alpha_1 - \alpha_0)}.$$

The sign of \tilde{p}_1 does not have any effect here.

Summarising, branches $w_i(\omega)$, $i = 1, 2, 3$ are stable in the low frequency regimes if the quantity

$$\tilde{p}_i \varepsilon^{\frac{1}{2}} (\alpha_1 - \alpha_0) - \tilde{\sigma}$$

is negative. Conversely, these branches become unstable if this quantity is positive.

High frequency expansions

The roots of the secular equation in the high frequency limit $\omega \rightarrow \infty$, may be obtained by putting $\omega = \zeta^{-1}$, so the secular equation (4.120) is written as

$$\tilde{F}(w) - \frac{i\zeta^{-1}\tilde{\sigma}w}{(1 - i\zeta^{-1}\alpha_0)(1 - i\zeta^{-1}\tau)} [w\varepsilon^{\frac{1}{2}}(1 - i\zeta^{-1}\tau)(\alpha_1 - \alpha_0) - \tilde{\sigma}] \tilde{G}(w) = 0. \quad (4.130)$$

Multiplying the denominator and numerator of the second term by ζ^2 we get

$$\tilde{F}(w) - \frac{i\tilde{\sigma}}{(\zeta - i\alpha_0)(\zeta - i\tau)} [w\varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0)(\zeta - i\tau) - \tilde{\sigma}\zeta] \tilde{G}(w) = 0, \quad (4.131)$$

and expanding this equation gives

$$\tilde{F}(w) - \frac{i\tilde{\sigma}w}{(\zeta - i\alpha_0)} w\varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0) \tilde{G}(w) + \frac{i\zeta\tilde{\sigma}^2}{(\zeta - i\alpha_0)(\zeta - i\tau)} \tilde{G}(w) = 0. \quad (4.132)$$

Putting $\zeta = 0$, we obtain

$$\tilde{H} \equiv \tilde{F}(w) + \frac{\tilde{\sigma}}{\alpha_0} w\varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0) \tilde{G}(w) = 0. \quad (4.133)$$

\tilde{H} is a quartic in w so there are four roots with one root $w = 0$ denoted by $\bar{h}_1 = 0$, and the other roots \bar{h}_2, \bar{h}_3 and \bar{h}_4 with $\bar{h}_2 < \bar{h}_3 < \bar{h}_4$. Now we want to examine the sign changes around these roots by using equation (4.133) and inequalities (4.118a)

we get, taking $\tilde{p}_1 > 0$ here,

$$\begin{aligned}
\tilde{H}(-\infty) &= -\infty < 0, \\
\tilde{H}(0) &= -\tilde{p}_1\tilde{p}_2\tilde{p}_3 + \tilde{\sigma}\alpha_0^{-1}(\alpha_1 - \alpha_0)\tilde{W}_1\tilde{W}_2, \\
\tilde{H}(\tilde{p}_1) &= \tilde{\sigma}\alpha_0^{-1}\tilde{p}_1\varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0)(\tilde{p}_1 - \tilde{W}_1)(\tilde{p}_1 - \tilde{W}_2) > 0, \\
\tilde{H}(\tilde{W}_1) &= \tilde{W}_1(\tilde{W}_1 - \tilde{p}_1)(\tilde{W}_1 - \tilde{p}_2)(\tilde{W}_1 - \tilde{p}_3) > 0, \\
\tilde{H}(\tilde{p}_2) &= \tilde{\sigma}\alpha_0^{-1}\tilde{p}_2\varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0)(\tilde{p}_2 - \tilde{W}_1)(\tilde{p}_2 - \tilde{W}_2) < 0, \\
\tilde{H}(\tilde{W}_2) &= \tilde{W}_2(\tilde{W}_2 - \tilde{p}_1)(\tilde{W}_2 - \tilde{p}_2)(\tilde{W}_2 - \tilde{p}_3) < 0, \\
\tilde{H}(\tilde{p}_3) &= \tilde{\sigma}\alpha_0^{-1}\tilde{p}_3\varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0)(\tilde{p}_3 - \tilde{W}_1)(\tilde{p}_3 - \tilde{W}_2) > 0, \\
\tilde{H}(\infty) &= \infty > 0.
\end{aligned}$$

From these inequalities we may determine the positions of the zeros of \tilde{H} , so that \bar{h}_2 is between zero and \tilde{p}_1 , \bar{h}_3 is between \tilde{W}_1 and \tilde{p}_2 and \bar{h}_4 is between \tilde{W}_2 and \tilde{p}_3 . Therefore, we get the same inequalities as (2.109b):

$$\bar{h}_2 < \tilde{p}_1 < \tilde{W}_1 < \bar{h}_3 < \tilde{p}_2 < \tilde{W}_2 < \bar{h}_4 < \tilde{p}_3.$$

Define a quartic polynomial

$$\bar{h}(w) = w(w - \bar{h}_2)(w - \bar{h}_3)(w - \bar{h}_4), \quad (4.134)$$

which must be a scalar multiple of \tilde{H} because both have the same four roots, so rewrite (4.134) as

$$\tilde{H}(w) = \bar{h}(w). \quad (4.135)$$

Now looking for roots when $\zeta \rightarrow 0$, let us rewrite (4.131) as

$$\tilde{F}(w) + \bar{K}\tilde{G}(w) = 0, \quad (4.136)$$

where

$$\bar{K} = \frac{-i\tilde{\sigma}}{(\zeta - i\alpha_0)(\zeta - i\tau)} [w\varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0)(\zeta - i\tau) - \tilde{\sigma}\zeta]. \quad (4.137)$$

Rewrite (4.137) as

$$\bar{K} = \frac{i\tilde{\sigma}}{\alpha_0\tau} \left[\zeta(w\varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0) - \tilde{\sigma}) - i\tau w\varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0) \right] \left(1 + \frac{i\zeta}{\alpha_0}\right)^{-1} \left(1 + \frac{i\zeta}{\tau}\right)^{-1}.$$

By using the binomial expansion we get

$$\bar{K} = \frac{i\tilde{\sigma}}{\alpha_0\tau} \left[\zeta \left(w\varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0) - \tilde{\sigma} \right) - i\tau w\varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0) \right] \left(1 - \frac{i\zeta}{\alpha_0} \right) \left(1 - \frac{i\zeta}{\tau} \right).$$

Expanding and ignoring high powers of ζ we obtain

$$\bar{K} = \frac{i\tilde{\sigma}}{\alpha_0\tau} \left[\zeta \left(w\varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0) - \tilde{\sigma} \right) - i\tau w\varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0) - \left(\frac{\tau\zeta}{\alpha_0} \right) w\varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0) - \zeta w\varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0) \right].$$

By simplifying we get

$$\bar{K} = \frac{-i\tilde{\sigma}}{\alpha_0\tau} \left[\zeta \left(w\varepsilon^{\frac{1}{2}}\tau\alpha_0^{-1}(\alpha_1 - \alpha_0) + \tilde{\sigma} \right) + i\tau w\varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0) \right]. \quad (4.138)$$

Inserting (4.138) into (4.136), the secular equation becomes

$$\tilde{F}(w) - \frac{i\tilde{\sigma}}{\alpha_0\tau} \left[\zeta \left(w\varepsilon^{\frac{1}{2}}\tau\alpha_0^{-1}(\alpha_1 - \alpha_0) + \tilde{\sigma} \right) + i\tau w\varepsilon^{\frac{1}{2}}(\alpha_1 - \alpha_0) \right] \tilde{G}(w) = 0. \quad (4.139)$$

By using (4.133) and (4.135), equation (4.139) may written as

$$\bar{h}(w) - \frac{i\zeta\tilde{\sigma}}{\alpha_0\tau} \left(w\varepsilon^{\frac{1}{2}}\tau\alpha_0^{-1}(\alpha_1 - \alpha_0) + \tilde{\sigma} \right) \tilde{G}(w) = 0. \quad (4.140)$$

This equation is a quartic in w provided that $\zeta > 0$, so there are four roots $w_i, i = 1, 2, 3, 4$. Power series expansion of the roots of the secular equation in the high frequency limit take the form

$$w_i(\zeta) = \bar{h}_i + \sum_{n=1}^{\infty} d_n^{(i)} (-i\zeta)^n, \quad i = 1, 2, 3, 4. \quad (4.141)$$

When $i = 1, n = 1$ we get

$$w_1(\zeta) = \bar{h}_1 + d_1^{(1)} (-i\zeta) + O(\zeta^2). \quad (4.142)$$

Substituting (4.142) into (4.140) we get

$$d_1^{(1)} = \left(\frac{\tilde{\sigma}^2}{\alpha_0\tau} \right) \frac{\tilde{W}_1\tilde{W}_2}{\bar{h}_2\bar{h}_3\bar{h}_4} > 0. \quad (4.143)$$

Inserting (4.143) into (4.142) we get

$$w_1(\zeta) = -i\zeta \left(\frac{\tilde{\sigma}^2}{\alpha_0\tau} \right) \frac{\tilde{G}(0)}{\bar{h}'(0)} + O(\zeta^2), \quad (4.144)$$

in which $\bar{h}_1 = 0$, so it is clear that $\text{Im } w_1(\zeta) < 0$, $w_1(\zeta)$ is stable.

When $i = 2, n = 1$ we get

$$w_2(\zeta) = \bar{h}_2 + d_1^{(2)}(-i\zeta) + O(\zeta^2). \quad (4.145)$$

Substituting (4.145) into (4.140) we obtain

$$d_1^{(2)} = \left(\frac{-\tilde{\sigma}}{\alpha_0 \tau} \right) \left(\bar{h}_2 \varepsilon^{\frac{1}{2}} \tau \alpha_0^{-1} (\alpha_1 - \alpha_0) + \tilde{\sigma} \right) \frac{(\bar{h}_2 - \tilde{W}_1)(\bar{h}_2 - \tilde{W}_2)}{\bar{h}_2(\bar{h}_2 - \bar{h}_3)(\bar{h}_2 - \bar{h}_4)}. \quad (4.146)$$

The stability condition is satisfied if $d_1^{(2)} > 0$, and $d_1^{(2)}$ is positive if

$$0 < \bar{h}_2 < \frac{\tilde{\sigma}}{\varepsilon^{\frac{1}{2}} \tau \alpha_0^{-1} (\alpha_1 - \alpha_0)},$$

and

$$\frac{\tilde{\sigma}}{\varepsilon^{\frac{1}{2}} \tau \alpha_0^{-1} (\alpha_1 - \alpha_0)} < \bar{h}_2 < 0.$$

Insert (4.146) into (4.145) to get

$$w_2(\zeta) = \bar{h}_2 + i\zeta \left(\frac{\tilde{\sigma}}{\alpha_0 \tau} \right) \left(\bar{h}_2 \varepsilon^{\frac{1}{2}} \tau \alpha_0^{-1} (\alpha_1 - \alpha_0) + \tilde{\sigma} \right) \frac{\tilde{G}(\bar{h}_2)}{\bar{h}'(\bar{h}_2)} + O(\zeta^2). \quad (4.147)$$

Similarly when $i = 3, 4, n = 1$ we obtain

$$w_3(\zeta) = \bar{h}_3 + i\zeta \left(\frac{\tilde{\sigma}}{\alpha_0 \tau} \right) \left(\bar{h}_3 \varepsilon^{\frac{1}{2}} \tau \alpha_0^{-1} (\alpha_1 - \alpha_0) + \tilde{\sigma} \right) \frac{\tilde{G}(\bar{h}_3)}{\bar{h}'(\bar{h}_3)} + O(\zeta^2), \quad (4.148)$$

$$w_4(\zeta) = \bar{h}_4 + i\zeta \left(\frac{\tilde{\sigma}}{\alpha_0 \tau} \right) \left(\bar{h}_4 \varepsilon^{\frac{1}{2}} \tau \alpha_0^{-1} (\alpha_1 - \alpha_0) + \tilde{\sigma} \right) \frac{\tilde{G}(\bar{h}_4)}{\bar{h}'(\bar{h}_4)} + O(\zeta^2). \quad (4.149)$$

Also, $w_3(\zeta)$ and $w_4(\zeta)$ are stable if

$$\bar{h}_3 < \frac{\tilde{\sigma}}{\varepsilon^{\frac{1}{2}} \tau \alpha_0^{-1} (\alpha_1 - \alpha_0)},$$

and

$$\bar{h}_4 < \frac{\tilde{\sigma}}{\varepsilon^{\frac{1}{2}} \tau \alpha_0^{-1} (\alpha_1 - \alpha_0)},$$

respectively.

Summarising, the stability of each branch is determined by the sign of

$$\bar{h}_i \varepsilon^{\frac{1}{2}} \tau \alpha_0^{-1} (\alpha_1 - \alpha_0) - \tilde{\sigma}, \quad i = 2, 3, 4,$$

stable if negative, unstable if positive.

Numerical results

In Figure 4.6 we have taken $\tilde{p}_1 > 0$. The branch $w_0(\omega)$ beginning at the origin is unstable for low frequency and stable for high frequency in each part of the Figure. All the other branches begin to the right of this branch. If α_0 and α_1 are small enough all the branches $w_i(\omega)$, $i = 1, 2, 3$, are stable for the low frequency and unstable for high frequency. This can be seen in the first subfigures of Figure 4.6 where α_0 and α_1 are small. As α_0 and α_1 increase, all branches $w_i(\omega)$, $i = 1, 2, 3$, become unstable in the low and high frequencies, see subfigure (f). It is clear that $w_0(\omega)$ retains instability in the low frequency and stability in the high frequency.

In Figure 4.7 we have taken $\tilde{p}_1 < 0$. The branch $w_1(\omega)$ beginning at $w = \tilde{p}_1$ is unstable in the low frequency and stable in the high frequency in each part of the Figure. All the other branches begin to the right of this branch are stable in the low frequency and unstable in the high frequency. Although, increasing of α_0 and α_1 all the branches w_i , $i = 0, 2, 3$ maintains stability in the low frequency and instability in the high frequency. Conversely, $w_1(\omega)$ maintains instability in the low frequency and stability in the high frequency.

In Figure 4.8 we illustrate the exceptional case $\tilde{p}_1 = 0$. Now two branches emanate from the origin, namely, $w_0(\omega)$ and $w_1(\omega)$, one unstable and the other stable in the low frequency but in the high frequency the situation is reversed, one with argument $-\pi/4$ and the other with argument $3\pi/4$. The other branches have similar situation as in Figure 4.7.

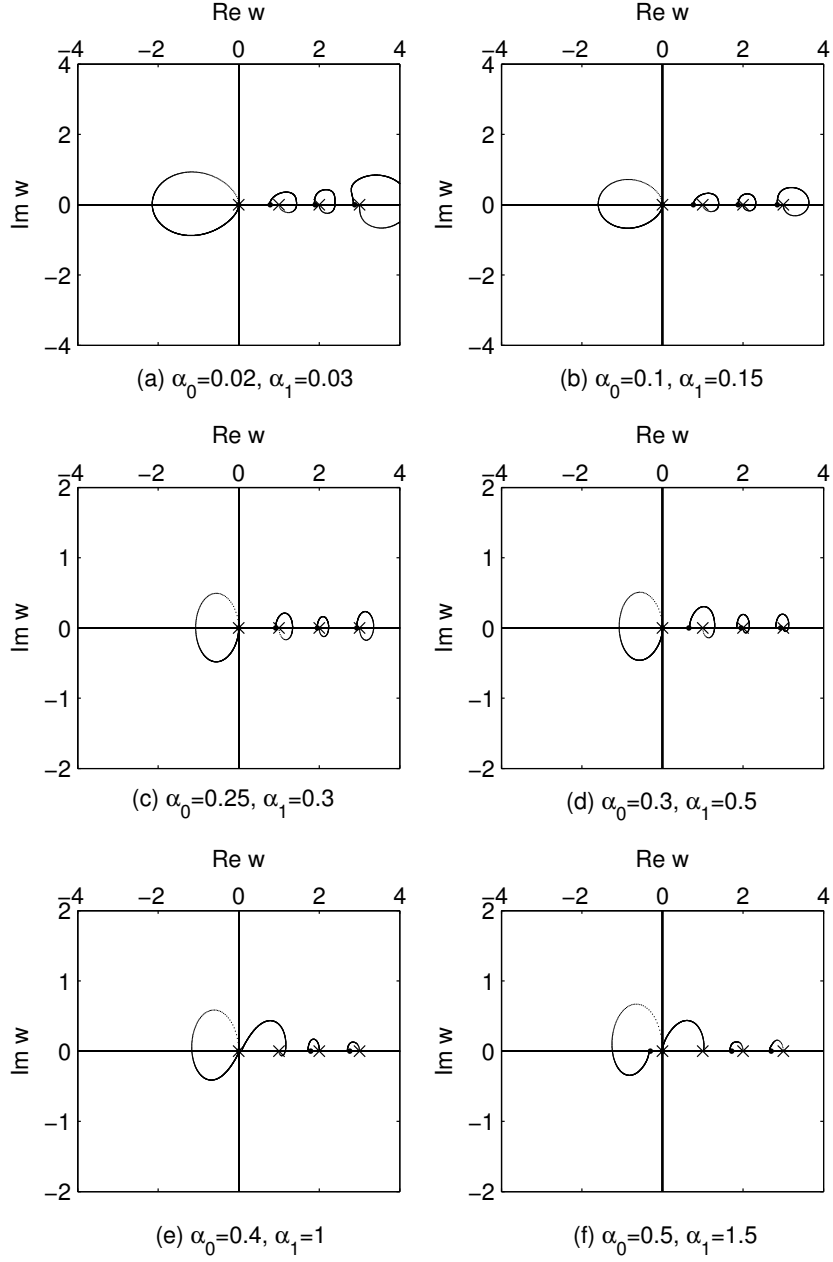


Figure 4.6: The longitudinal squared wave speeds of constrained anisotropic TRDTE+GTE (2). For each part, $\tilde{p}_1 = 1, \tilde{p}_2 = 2, \tilde{p}_3 = 3, \tilde{W}_1 = 1.5, \tilde{W}_2 = 2.5, \tilde{\sigma} = 1, \varepsilon = 1$.

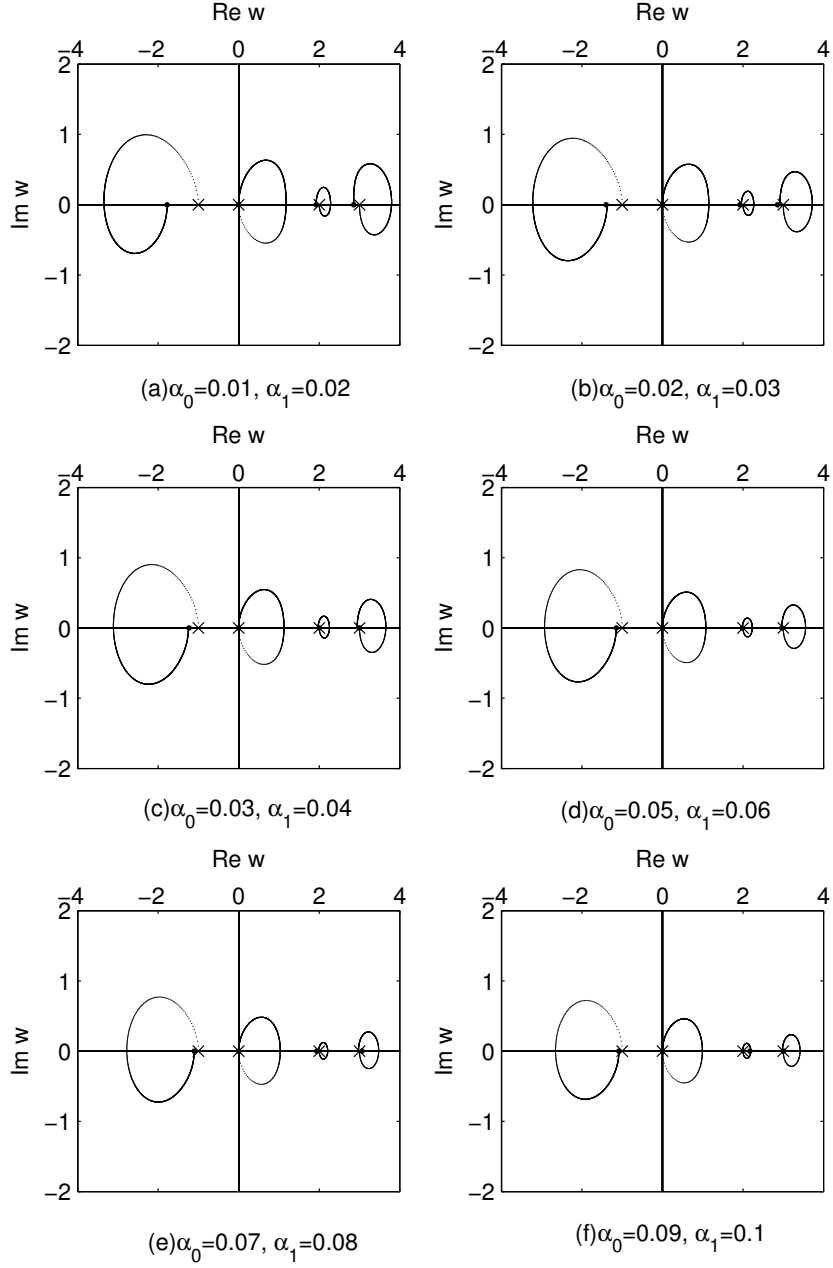


Figure 4.7: The longitudinal squared wave speeds of constrained anisotropic TRDTE+GTE (2). For each part, $\tilde{p}_1 = -1, \tilde{p}_2 = 2, \tilde{p}_3 = 3, \tilde{W}_1 = 1.5, \tilde{W}_2 = 2.5, \tilde{\sigma} = 1, \varepsilon = 1$.

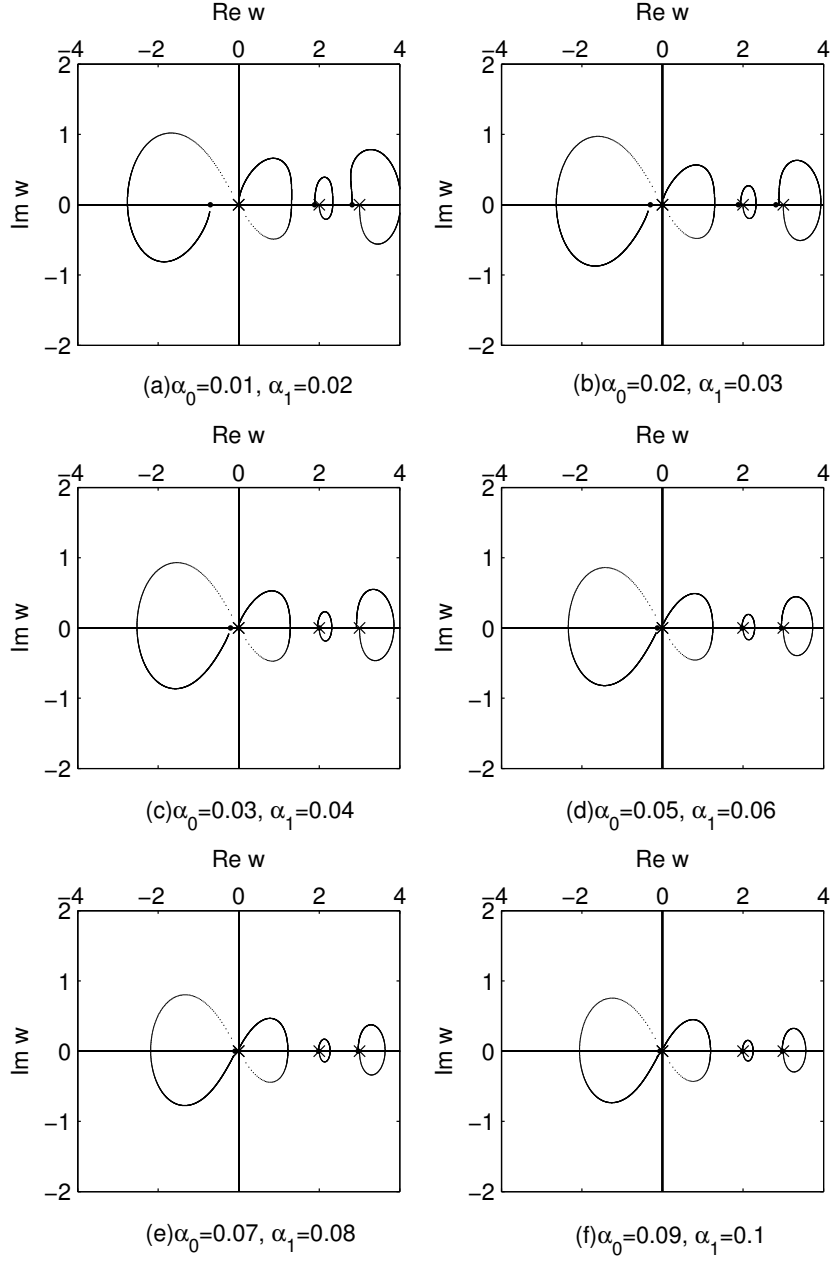


Figure 4.8: The longitudinal squared wave speeds of constrained anisotropic TRDTE+GTE (2). For each part, $\tilde{p}_1 = 0, \tilde{p}_2 = 2, \tilde{p}_3 = 3, \tilde{W}_1 = 1.5, \tilde{W}_2 = 2.5, \tilde{\sigma} = 1, \varepsilon = 1$.

4.4 Constrained isotropic TRDTE+GTE (2)

4.4.1 Usual form of deformation-temperature constraint

In this section we apply (2.191) to equations (4.67), (4.70) and the linearised form of the deformation temperature constraint (4.65) to get the field equations of Ignaczak's theory for an isotropic material incompressible with the usual deformation-temperature constraint.

$$\left. \begin{aligned} (\tilde{\lambda} + \tilde{\mu})u_{j,ij} + \tilde{\mu}u_{i,jj} - \beta(\theta + \alpha_1\dot{\theta})_{,i} + \tilde{N}\tilde{\eta}_{,i} &= \rho\ddot{u}_i, \\ k\theta_{,ii} - T\beta(\dot{u}_{i,i} + \tau\ddot{u}_{i,i}) - \rho c(\dot{\theta} + (\alpha_0 + \tau)\ddot{\theta} + \alpha_0\tau\ddot{\ddot{\theta}}) - \alpha T(\dot{\tilde{\eta}} + \tau\ddot{\tilde{\eta}}) &= 0, \\ \tilde{N}u_{i,i} - \alpha\theta &= 0. \end{aligned} \right\} \quad (4.150)$$

The secular equation

Now we are seeking solutions of (4.150) in the form of plane harmonic waves (2.86) with aid (3.159a), exactly similar to previous sections. So, we get the following system of algebraic equations

$$\left. \begin{aligned} [(\tilde{\mu} - \rho s^{-2})\delta_{ij} + (\tilde{\lambda} + \tilde{\mu})n_i n_j]U_i + i\beta(\omega s)^{-1}n_i(1 - i\omega\alpha_1)\Theta - i\tilde{N}n_j(\omega s)^{-1}\tilde{H} &= 0, \\ T\beta\omega s(1 - i\omega\tau)n_i U_i + (\omega s^2 k - i\rho c(1 - i\omega\alpha_0)(1 - i\omega\tau))\Theta - i\alpha T(1 - i\omega\tau)\tilde{H} &= 0, \\ \tilde{N}n_j i\omega s U_i - \alpha\Theta &= 0. \end{aligned} \right\} \quad (4.151)$$

We will now eliminate Θ and \tilde{H} between (4.151). From (4.151)₃ we can write Θ as follows

$$\Theta = \frac{i\omega s \tilde{N} n_j U_j}{\alpha}. \quad (4.152)$$

Substituting (4.152) into (4.151)₂, we get

$$\tilde{H} = -i\alpha^{-1}\beta\omega s n_i U_i + \left(\frac{\omega s \tilde{N} n_j}{\alpha^2 T(1 - i\omega\tau)} \right) (\omega s^2 k - i\rho c(1 - i\omega\alpha_0)(1 - i\omega\tau)) U_i. \quad (4.153)$$

Inserting (4.152) and (4.153) into (4.151)₁, we get

$$\begin{aligned} \left[(\tilde{\mu} - w)\delta_{ij} + (\tilde{\lambda} + \tilde{\mu})n_i n_j \right] U_j + i\beta(\omega s)^{-1}n_i(1 - i\omega\alpha_1) \left(\frac{i\omega s \tilde{N} n_j U_j}{\alpha} \right) - i\tilde{N}(\omega s)^{-1}n_i \\ \left[-i\alpha^{-1}\beta\omega s n_j U_j + \frac{\omega s \tilde{N} n_j}{\alpha^2 T(1 - i\omega\tau)} (\omega s^2 k - i\rho c(1 - i\omega\alpha_0)(1 - i\omega\tau)) U_j \right] = 0. \end{aligned}$$

After simplifying and rearranging this equation we obtain

$$\left\{ (\tilde{\mu} - w)\delta_{ij} + \left[\tilde{\lambda} + \tilde{\mu} - \alpha^{-1}\beta\tilde{N}(2 - i\omega\alpha_1) - \left(\frac{\rho c\tilde{N}^2}{\alpha^2 T(1 - i\omega\tau)} \right) \right. \right. \\ \left. \left. \left((1 - i\omega\alpha_0)(1 - i\omega\tau) + \frac{i\omega k}{wc} \right) \right] n_i n_j \right\} U_j = 0, \quad (4.154)$$

which gives in direct notation the secular equation

$$\det \left\{ (\tilde{\mu} - w)\mathbf{1} + \left[\tilde{\lambda} + \tilde{\mu} - \alpha^{-1}\beta\tilde{N}(2 - i\omega\alpha_1) - \left(\frac{\rho c\tilde{N}^2}{\alpha^2 T(1 - i\omega\tau)} \right) \right. \right. \\ \left. \left. \left((1 - i\omega\alpha_0)(1 - i\omega\tau) + \frac{i\omega k}{wc} \right) \right] \mathbf{n} \otimes \mathbf{n} \right\} = 0. \quad (4.155)$$

Non-dimensionalize this equation by applying the dimensionless quantities (2.57) and further dimensionless quantities

$$\alpha' = \alpha T, \quad c' = \rho c T / \gamma, \quad \omega^* = \gamma c / k$$

to get

$$\det \left\{ (\tilde{\mu}' - w')\mathbf{1} + \left[\tilde{\lambda}' + \tilde{\mu}' - \frac{(\varepsilon c')^{1/2}\tilde{N}}{\alpha'}(2 - i\omega'\alpha'_1) - \left(\frac{c'\tilde{N}^2}{\alpha'^2(1 - i\omega'\tau')} \right) \right. \right. \\ \left. \left. \left((1 - i\omega'\alpha'_0)(1 - i\omega'\tau') + \frac{i\omega'}{w'} \right) \right] \mathbf{n} \otimes \mathbf{n} \right\} = 0. \quad (4.156)$$

Now by using the standard identity (2.60), dropping the dashes for convenience, we get the secular equation as follows

$$(w - \tilde{\mu})^2 \left\{ w^2 - w \left[1 - (\varepsilon c)^{1/2}\tilde{N}\alpha^{-1}(2 - i\omega\alpha_1) - c\tilde{N}^2\alpha^{-2}(1 - i\omega\alpha_0) \right] \right. \\ \left. + \frac{i\omega c\tilde{N}^2}{\alpha^2(1 - i\omega\tau)} \right\} = 0. \quad (4.157)$$

This is the secular equation for isotropic TRDTE+GTE (2) which is constrained by the usual deformation temperature constraint and has not previously appeared in the literature.

The repeated root $w = \tilde{\mu}$ represents two purely elastic transverse waves, and longitudinal waves are represented by roots of the following quadratic equation

$$\alpha^2 w^2 - w \left\{ \alpha^2 - (\varepsilon c)^{1/2}\alpha\tilde{N}(2 - i\omega\alpha_1) - c\tilde{N}^2(1 - i\omega\alpha_0) \right\} + \frac{i\omega c\tilde{N}^2}{(1 - i\omega\tau)} = 0. \quad (4.158)$$

Equation (4.158) may be written in dimensionless form as

$$w^2 - w \left\{ 1 - \varepsilon^{1/2} \tilde{\sigma} (2 - i\omega\alpha_1) - \tilde{\sigma}^2 (1 - i\omega\alpha_0) \right\} + \frac{i\omega\tilde{\sigma}^2}{(1 - i\omega\tau)} = 0. \quad (4.159)$$

This equation is rewritten as

$$w^2(1 - i\omega\tau) - w \left\{ (1 - i\omega\tau) - \varepsilon^{1/2} \tilde{\sigma} (2 - i\omega\alpha_1)(1 - i\omega\tau) - \tilde{\sigma}^2 (1 - i\omega\alpha_0)(1 - i\omega\tau) \right\} + i\omega\tilde{\sigma}^2 = 0, \quad (4.159a)$$

where $\tilde{\sigma}$ is defined earlier in (2.202). On putting $\tau = 0$ in (4.159a) we get the corresponding secular equation (2.201) of TRDTE, as expected.

Recall the special cases that were discussed in Section 2.4.1.

Case 1: The isothermal constraint ($\tilde{N} = 0, \alpha \neq 0$)

Putting $\tilde{N} = 0$ in equation (4.158) gives the quadratic equation with the same results as (2.195):

$$w_1 = 0, \quad w_2 = 1.$$

Case 2: The purely mechanical constraint ($\tilde{N} \neq 0, \alpha = 0$)

Inserting $\alpha = 0$ into equation (4.158) gives the single branch

$$w = \frac{-i\omega}{(1 - i\omega\alpha_0)(1 - i\omega\tau)}. \quad (4.160)$$

Equation (4.160) is purely diffusive and also satisfies the stability condition $\text{Im } w \leq 0$. But in the general case, in which neither \tilde{N} nor α is equal to zero, it is convenient to go back to equation (4.159). The roots of this equation are

$$w_{1,2} = \bar{A} \pm \left[\bar{A}^2 - \frac{i\omega\tilde{\sigma}^2}{(1 - i\omega\tau)} \right]^{\frac{1}{2}}, \quad (4.161)$$

where

$$\bar{A} = \frac{1}{2} \left[1 - \varepsilon^{1/2} \tilde{\sigma} (2 - i\omega\alpha_1) - \tilde{\sigma}^2 (1 - i\omega\alpha_0) \right]. \quad (4.162)$$

This equation may be written as

$$\bar{A} = \frac{1}{2} \left[1 - \varepsilon^{1/2} \tilde{\sigma} - \tilde{\sigma}^2 + i\omega\tilde{\sigma}(\varepsilon^{1/2}\alpha_1 + \alpha_0\tilde{\sigma}) \right].$$

For fixed $\varepsilon \geq 0$, as $\tilde{\sigma}$ increases from 0 to ∞ , $\text{Re } \bar{A}$ at $\omega = 0$ decreases from $\frac{1}{2}$ to $-\infty$. $\text{Re } \bar{A}$ becomes 0 at $\omega = 0$ for a critical value of $\tilde{\sigma}$ given by

$$\tilde{\sigma}_c = (1 + \varepsilon)^{1/2} - \varepsilon^{1/2}. \quad (4.163)$$

In the special case where $\tilde{\sigma} = \tilde{\sigma}_c$, so $\text{Re } \bar{A} = 0$ at $\omega = 0$ in (4.161), we get the following form

$$w = \pm e^{i\frac{\pi}{4}} \omega^{\frac{1}{2}} \tilde{\sigma}_c + O(\omega). \quad (4.164)$$

Low frequency expansions

The roots of the secular equation (4.159a) at $\omega = 0$ are

$$w_1 = 1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2, \quad \text{and} \quad w_2 = 0.$$

But the roots as $\omega \rightarrow 0$ take the form

$$w_1 = 1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2 + A(i\omega) + O(\omega^2), \quad \text{and} \quad w_2 = B(i\omega) + O(\omega^2). \quad (4.164a)$$

Inserting (4.164a) into (4.159a) we obtain the roots as

$$w_1 = 1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2 + i\omega\tilde{\sigma}\left\{\varepsilon^{1/2}\alpha_1 + \alpha_0\tilde{\sigma} - \frac{\tilde{\sigma}}{1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2}\right\} + O(\omega^2),$$

and

$$w_2 = \frac{i\omega\tilde{\sigma}^2}{1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2} + O(\omega^2).$$

If $\tilde{\sigma} > \tilde{\sigma}_c$ then $\text{Im } w_1 > 0$, then w_1 is unstable, if $\tilde{\sigma} < \tilde{\sigma}_c$ we cannot tell the sign of $\text{Im } w_1(\omega)$ because it depends on the relative values of the quantities occurring. But it is clear that if $\tilde{\sigma} > \tilde{\sigma}_c$, $\text{Im } w_2(\omega) < 0$, so w_2 is stable and if $\tilde{\sigma} < \tilde{\sigma}_c$, $\text{Im } w_2(\omega) > 0$, thus w_2 is unstable. If $\tilde{\sigma} = \tilde{\sigma}_c$ the analysis is not valid and we return to the roots in the special case (4.164).

High frequency expansions

In the high frequency limit as $\omega \rightarrow \infty$, i.e. $(\omega)^{-1} \rightarrow 0$, the secular equation (4.159a), after dividing by $(i\omega)^2$, becomes

$$\begin{aligned} & w^2[(i\omega)^{-2} - (i\omega)^{-1}\tau] - \\ & w\left\{(i\omega)^{-2} - (i\omega)^{-1}\tau - \varepsilon^{1/2}\tilde{\sigma}\left[2(i\omega)^{-2} - (i\omega)^{-1}(\alpha_1 + 2\tau) + \alpha_1\tau\right] - \tilde{\sigma}^2\left[(i\omega)^{-2} - (i\omega)^{-1}(\alpha_0 + \tau) + \alpha_0\tau\right]\right\} \\ & + (i\omega)^{-1}\tilde{\sigma}^2 = 0. \end{aligned} \quad (4.164b)$$

Putting $(i\omega)^{-1} = 0$ we get the roots of the secular equation as

$$w_1 = 0, \quad \text{and} \quad w_2 \rightarrow \infty.$$

Now look for the roots as $(i\omega)^{-1} \rightarrow 0$; the roots take the form

$$w_1 = A(i\omega)^{-1} + O(\omega^{-2}), \quad \text{and} \quad w_2 = B(i\omega) + C + O(\omega^{-1}). \quad (4.164c)$$

Inserting (4.164c) into (4.164b) we obtain

$$w_1 = \frac{i\omega^{-1}\tilde{\sigma}}{\tau(\varepsilon^{1/2}\alpha_1 + \tilde{\sigma}\alpha_0)} + O(\omega^{-2}),$$

and

$$w_2 = i\omega\tilde{\sigma}(\varepsilon^{1/2}\alpha_1 + \tilde{\sigma}\alpha_0) + (1 - \varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2) + O(\omega^{-1}).$$

It is clear that $\text{Im } w_1(\omega) > 0$ and $\text{Im } w_2(\omega) > 0$, so w_1 and w_2 are unstable in the high frequency limit.

Now we consider the two previous special cases.

Case 1: The isothermal constraint viewed as the limit $\tilde{\sigma} \rightarrow 0$

The roots of equation (4.161) are

$$w_{1,2} = \frac{1}{2} \left[1 - \varepsilon^{1/2}\tilde{\sigma}(2 - i\omega\alpha_1) - \tilde{\sigma}^2(1 - i\omega\alpha_0) \right] \pm \frac{1}{2} \left\{ \left[1 - \varepsilon^{1/2}\tilde{\sigma}(2 - i\omega\alpha_1) - \tilde{\sigma}^2(1 - i\omega\alpha_0) \right]^2 - \frac{4i\omega\tilde{\sigma}^2}{(1 - i\omega\tau)} \right\}^{1/2}. \quad (4.164d)$$

After expanding and using the binomial expansion we get

$$w_1 = 1 - \varepsilon^{1/2}\tilde{\sigma}(2 - i\omega\alpha_1) - \tilde{\sigma}^2(1 - i\omega\alpha_0) - \frac{i\omega\tilde{\sigma}^2}{(1 - i\omega\tau)} + O(\tilde{\sigma}^3), \quad (4.165)$$

$$w_2 = \frac{i\omega\tilde{\sigma}^2}{(1 - i\omega\tau)} + O(\tilde{\sigma}^3). \quad (4.166)$$

Case 2: The purely mechanical constraint viewed as the limit $\tilde{\sigma} \rightarrow \infty$

When $\tilde{\sigma} \rightarrow \infty$, $\frac{1}{\tilde{\sigma}}$ is small, and from (4.164d), after expanding and factorising by $\tilde{\sigma}^4(1 - i\omega\alpha_0)^2$, then using the binomial expansion, we obtain

$$w_1 = \frac{-i\omega}{(1 - i\omega\alpha_0)(1 - i\omega\tau)} \left[1 - \varepsilon^{1/2}\tilde{\sigma}^{-1} \frac{(2 - i\omega\alpha_1)}{(1 - i\omega\alpha_0)} \right] + O(\tilde{\sigma}^{-2}), \quad (4.167)$$

$$w_2 = 1 - \varepsilon^{1/2}\tilde{\sigma}(2 - i\omega\alpha_1) - \tilde{\sigma}^2(1 - i\omega\alpha_0) + \frac{i\omega}{(1 - i\omega\alpha_0)(1 - i\omega\tau)} \left[1 - \varepsilon^{1/2}\tilde{\sigma}^{-1} \frac{(2 - i\omega\alpha_1)}{(1 - i\omega\alpha_0)} \right] + O(\tilde{\sigma}^{-2}). \quad (4.168)$$

Numerical results

Figure 4.9 illustrates two longitudinal waves one branch starting and ending at origin and is stable in the low frequency and unstable in the high frequency. The other branch starting from the point $1 - \varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2$ and tending to infinity and it retains the instability in the low and high frequencies.

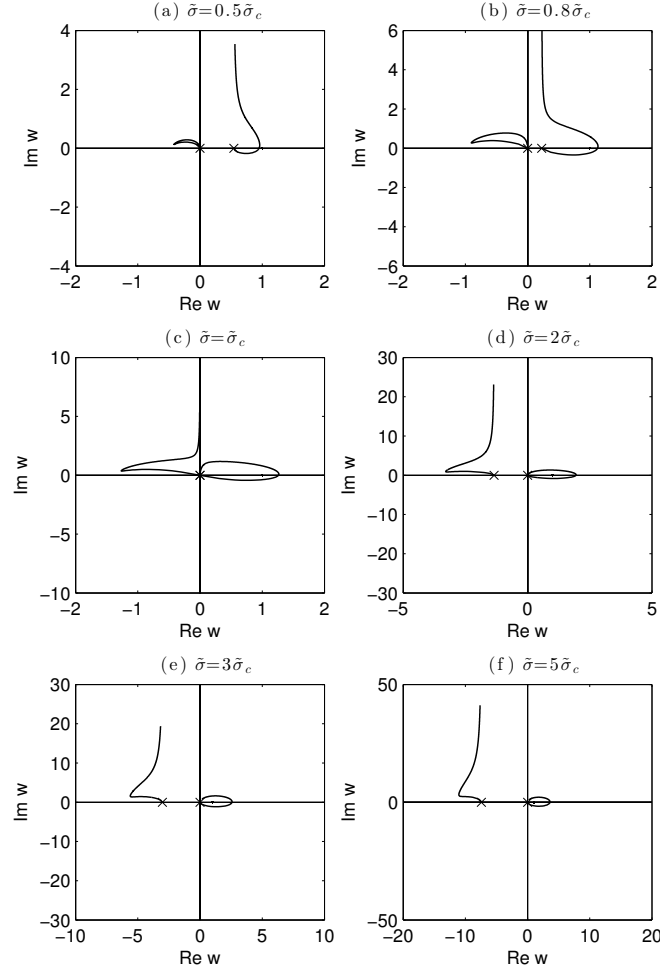


Figure 4.9: The longitudinal squared wave speeds of isotropic thermelastic material for Ignaczak's theory with incompressibility at uniform temperature. For each part ($\varepsilon = 1, \alpha_0 = 0.01, \alpha_1 = 0.02, \tau = 0.1$), (a) $\tilde{\sigma} = 0.5\tilde{\sigma}_c$, (b) $\tilde{\sigma} = 0.8\tilde{\sigma}_c$, (c) $\tilde{\sigma} = \tilde{\sigma}_c$, (d) $\tilde{\sigma} = 2\tilde{\sigma}_c$, (e) $\tilde{\sigma} = 3\tilde{\sigma}_c$, (f) $\tilde{\sigma} = 5\tilde{\sigma}_c$.

4.4.2 Alternative form of deformation-temperature constraints

The field equations for Ignaczak's theory, see [25], of constrained isotropic thermoe-
lastic materials are (4.150)_{1,2} with the linearised alternative form of deformation-
temperature constraint (2.144)

$$\left. \begin{aligned} (\tilde{\lambda} + \tilde{\mu})u_{j,ij} + \tilde{\mu}u_{i,jj} - \beta(\theta + \alpha_1\dot{\theta})_{,i} + \tilde{N}\tilde{\eta}_{,i} &= \rho\ddot{u}_i, \\ k\theta_{,ii} - T\beta(\dot{u}_{j,j} + \tau\ddot{u}_{j,j}) - \rho c(\dot{\theta} + (\alpha_0 + \tau)\ddot{\theta} + \alpha_0\tau\ddot{\ddot{\theta}}) - \alpha T(\dot{\tilde{\eta}} + \tau\ddot{\tilde{\eta}}) &= 0, \\ \tilde{N}u_{i,i} - \alpha(\theta + \alpha_0\dot{\theta}) &= 0. \end{aligned} \right\} \quad (4.169)$$

Now we are looking for solutions in the form of plane harmonic waves (2.86), and by
inserting (2.86) into (4.169) we get the system of algebraic equations

$$\left. \begin{aligned} [(\tilde{\mu} - \rho s^{-2})\delta_{ij} + (\tilde{\lambda} + \tilde{\mu})n_in_j]U_i + i\beta(\omega s)^{-1}n_i(1 - i\omega\alpha_1)\Theta - i\tilde{N}n_j(\omega s)^{-1}\tilde{H} &= 0, \\ T\beta\omega s(1 - i\omega\tau)n_iU_i + (\omega s^2k - i\rho c(1 - i\omega\alpha_0)(1 - i\omega\tau))\Theta - i\alpha T(1 - i\omega\tau)\tilde{H} &= 0, \\ \tilde{N}n_i i\omega s U_i - \alpha(1 - i\omega\alpha_0)\Theta &= 0. \end{aligned} \right\} \quad (4.170)$$

We will now eliminate Θ and \tilde{H} between (4.170). From (4.170)₃ we can rewrite Θ as
follows

$$\Theta = \frac{i\omega s\tilde{N}n_jU_j}{\alpha(1 - i\omega\alpha_0)}. \quad (4.171)$$

Substituting (4.171) into (4.170)₃, we get

$$\tilde{H} = -i\alpha^{-1}\beta\omega s n_i U_i + \left(\frac{\omega s\tilde{N}n_j}{\alpha^2 T(1 - i\omega\tau)(1 - i\omega\alpha_0)} \right) [\omega s^2k - i\rho c(1 - i\omega\alpha_0)(1 - i\omega\tau)] U_j. \quad (4.172)$$

Inserting (4.171) and (4.172) into (4.170)₁, we get

$$\left[(\tilde{\mu} - w)\delta_{ij} + (\tilde{\lambda} + \tilde{\mu})n_in_j \right] U_j + i\beta(\omega s)^{-1}n_i(1 - i\omega\alpha_1) \left(\frac{i\omega s\tilde{N}n_jU_j}{\alpha(1 - i\omega\alpha_0)} \right) - i\tilde{N}(\omega s)^{-1}n_i \\ \left[-i\alpha^{-1}\beta\omega s n_j U_j + \left(\frac{\omega s\tilde{N}n_j}{\alpha^2 T(1 - i\omega\tau)(1 - i\omega\alpha_0)} \right) (\omega s^2k - i\rho c(1 - i\omega\alpha_0)(1 - i\omega\tau)) U_j \right] = 0. \quad (4.173)$$

After simplifying and rearranging the equation we obtain

$$\left\{ (\tilde{\mu} - w)\delta_{ij} + \left[\tilde{\lambda} + \tilde{\mu} - \alpha^{-1}\beta\tilde{N} \left(1 + \frac{1 - i\omega\alpha_1}{1 - i\omega\alpha_0} \right) - \left(\frac{\rho c\tilde{N}^2}{\alpha^2 T(1 - i\omega\tau)(1 - i\omega\alpha_0)} \right) \right. \right. \\ \left. \left. \left[(1 - i\omega\alpha_0)(1 - i\omega\tau) + \frac{i\omega k}{wc} \right] n_in_j \right\} U_j = 0, \quad (4.174)$$

which gives in direct notation the secular equation

$$\det \left\{ (\tilde{\mu} - w)\mathbf{1} + \left[\tilde{\lambda} + \tilde{\mu} - \alpha^{-1}\beta\tilde{N} \left(1 + \frac{1 - i\omega\alpha_1}{1 - i\omega\alpha_0} \right) - \left(\frac{\rho c \tilde{N}^2}{\alpha^2 T (1 - i\omega\tau)(1 - i\omega\alpha_0)} \right) \right. \right. \\ \left. \left. \left[(1 - i\omega\alpha_0)(1 - i\omega\tau) + \frac{i\omega k}{wc} \right] \right] \mathbf{n} \otimes \mathbf{n} \right\} = 0. \quad (4.175)$$

Non-dimensionalising this equation by applying the dimensionless quantities (2.57) we get

$$\det \left\{ (\tilde{\mu}' - w')\mathbf{1} + \left[\tilde{\lambda}' + \tilde{\mu}' - \frac{\tilde{N}(\varepsilon c')^{1/2}}{\alpha'} \left(1 + \frac{1 - i\omega'\alpha'_1}{1 - i\omega'\alpha'_0} \right) \right] - \left(\frac{c' \tilde{N}^2}{\alpha'^2 (1 - i\omega'\tau')(1 - i\omega'\alpha'_0)} \right) \right. \\ \left. \left[(1 - i\omega'\alpha'_0)(1 - i\omega'\tau') + \frac{i\omega'}{w'} \right] \mathbf{n} \otimes \mathbf{n} \right\} = 0. \quad (4.176)$$

Now by using the standard identity (2.60), dropping the dashes for convenience, we get the secular equation as follows

$$(w - \tilde{\mu})^2 \left\{ w^2 - w \left[1 - (\varepsilon c)^{1/2} \alpha^{-1} \tilde{N} \left(1 + \frac{1 - i\omega\alpha_1}{1 - i\omega\alpha_0} \right) - \frac{c \tilde{N}^2}{\alpha^2} \right] + \left(\frac{i\omega c \tilde{N}^2}{\alpha^2 (1 - i\omega\tau)(1 - i\omega\alpha_0)} \right) \right\} = 0. \quad (4.177)$$

This is the secular equation for isotropic TRDTE+GTE (2) which is constrained by the alternative deformation temperature constraint and has not previously appeared in the literature.

The repeated root $w = \tilde{\mu}$ represents two purely elastic transverse waves and the longitudinal waves are represented by the following quadratic equation

$$w^2 - w \left[1 - (\varepsilon c)^{1/2} \alpha^{-1} \tilde{N} \left(1 + \frac{1 - i\omega\alpha_1}{1 - i\omega\alpha_0} \right) - \frac{c \tilde{N}^2}{\alpha^2} \right] + \left(\frac{i\omega c \tilde{N}^2}{\alpha^2 (1 - i\omega\tau)(1 - i\omega\alpha_0)} \right) = 0. \quad (4.178)$$

We can rewrite equation (4.178) as

$$w^2 + w(\tilde{\sigma}^2 + \varepsilon^{1/2} \tilde{\sigma} (1 + \frac{1 - i\omega\alpha_1}{1 - i\omega\alpha_0}) - 1) + i\omega \tilde{\sigma}^2 [(1 - i\omega\tau)(1 - i\omega\alpha_0)]^{-1} = 0, \quad (4.179)$$

where $\tilde{\sigma}$ is defined in (2.202).

The roots of this quadratic equation we can get similarly to previous constrained isotropic sections.

$$w_{1,2} = \bar{A} \pm \left[\bar{A}^2 - \frac{i\omega \tilde{\sigma}^2}{(1 - i\omega\tau)(1 - i\omega\alpha_0)} \right]^{\frac{1}{2}}, \quad (4.180)$$

where

$$\bar{A} = \frac{1}{2} \left[1 - \varepsilon^{1/2} \tilde{\sigma} \left(1 + \frac{1 - i\omega\alpha_1}{1 - i\omega\alpha_0} \right) - \tilde{\sigma}^2 \right]. \quad (4.181)$$

Equation (4.181) may be rewritten as

$$\bar{A} = \frac{1}{2} \left\{ 1 - \frac{\varepsilon^{1/2} \tilde{\sigma}}{1 + \omega^2 \alpha_0^2} [2 + \omega^2 \alpha_0 (\alpha_1 + \alpha_0)] - \tilde{\sigma}^2 + \frac{i\omega \varepsilon^{1/2} \tilde{\sigma}}{1 + \omega^2 \alpha_0^2} (\alpha_1 - \alpha_0) \right\}.$$

For fixed $\varepsilon \geq 0$, as $\tilde{\sigma}$ increases from 0 to ∞ , $\text{Re } \bar{A}$ at $\omega = 0$ decreases from $\frac{1}{2}$ to $-\infty$. $\text{Re } \bar{A}$ at $\omega = 0$ becomes 0 for a critical value of $\tilde{\sigma}$ given by

$$\tilde{\sigma}_c = (1 + \varepsilon)^{1/2} - \varepsilon^{1/2}. \quad (4.182)$$

In the special case where $\tilde{\sigma} = \tilde{\sigma}_c$, so $\text{Re } \bar{A} = 0$ at $\omega = 0$ in (4.180), we get the following form

$$w = \pm e^{i\frac{\pi}{4}} \omega^{\frac{1}{2}} \tilde{\sigma}_c + O(\omega). \quad (4.183)$$

Low frequency expansions

The secular equation (4.179) after expanding may be written as

$$\begin{aligned} & w^2 [1 - i\omega(\tau + \alpha_0) + (i\omega)^2 \tau \alpha_0] - \\ & w \left\{ [1 - i\omega(\tau + \alpha_0) + (i\omega)^2 \tau \alpha_0] - \varepsilon^{1/2} \tilde{\sigma} [2 - i\omega(\alpha_1 + \alpha_0 + 2\tau) + (i\omega)^2 \tau (\alpha_1 + \alpha_0)] \right. \\ & \quad \left. - \tilde{\sigma}^2 [1 - i\omega(\tau + \alpha_0) + (i\omega)^2 \tau \alpha_0] \right\} + i\omega \tilde{\sigma}^2 = 0. \end{aligned} \quad (4.182a)$$

In the low frequency expansions at $\omega = 0$ the roots of the secular equation (4.182a) are

$$w_1 = 1 - 2\varepsilon^{1/2} \tilde{\sigma} - \tilde{\sigma}^2, \quad \text{and} \quad w_2 = 0.$$

As $\omega \rightarrow 0$ the roots take the form

$$w_1 = 1 - 2\varepsilon^{1/2} \tilde{\sigma} - \tilde{\sigma}^2 + A(i\omega) + O(\omega^2), \quad \text{and} \quad w_2 = B(i\omega) + O(\omega^2). \quad (4.182b)$$

Inserting (4.182b) into (4.182a) we get

$$w_1 = 1 - 2\varepsilon^{1/2} \tilde{\sigma} - \tilde{\sigma}^2 + i\omega \tilde{\sigma} \left\{ \varepsilon^{1/2} (\alpha_1 - \alpha_0) - \frac{\tilde{\sigma}}{1 - 2\varepsilon^{1/2} \tilde{\sigma} - \tilde{\sigma}^2} \right\} + O(\omega^2),$$

and

$$w_2 = \frac{i\omega \tilde{\sigma}^2}{1 - 2\varepsilon^{1/2} \tilde{\sigma} - \tilde{\sigma}^2} + O(\omega^2).$$

If $\tilde{\sigma} > \tilde{\sigma}_c$ $\text{Im } w_1(\omega) > 0$ so w_1 is unstable and if $\tilde{\sigma} < \tilde{\sigma}_c$ we cannot tell the sign of $\text{Im } w_1(\omega)$. If $\tilde{\sigma} > \tilde{\sigma}_c$, $\text{Im } w_2(\omega) < 0$, thus w_2 is stable and if $\tilde{\sigma} < \tilde{\sigma}_c$, $\text{Im } w_2(\omega) > 0$, thus w_2 is unstable. If $\tilde{\sigma} = \tilde{\sigma}_c$ the analysis is not valid and we return to roots in the special case (4.183).

High frequency expansions

In the high frequency limits $\omega \rightarrow \infty$, i.e. $(\omega)^{-1} \rightarrow 0$. Equation (4.182a), after dividing by $(i\omega)^2$, becomes

$$\begin{aligned} & w^2[(i\omega)^{-2} - (i\omega)^{-1}(\tau + \alpha_0) - \tau\alpha_0] - \\ & w \left\{ [(i\omega)^{-2} - (i\omega)^{-1}(\tau + \alpha_0) - \tau\alpha_0] - \varepsilon^{1/2}\tilde{\sigma}[2(i\omega)^{-2} - (i\omega)^{-1}(\alpha_1 + \alpha_0 + 2\tau) - \tau(\alpha_1 + \alpha_0)] \right. \\ & \quad \left. - \tilde{\sigma}^2[(i\omega)^{-2} - (i\omega)^{-1}(\tau + \alpha_0) - \tau\alpha_0] \right\} + (i\omega)^{-1}\sigma^2 = 0. \end{aligned} \quad (4.182c)$$

Putting $(i\omega)^{-1} = 0$ we get

$$w_1 = 1 - \varepsilon^{1/2}\tilde{\sigma}(1 + \frac{\alpha_1}{\alpha_0}) - \tilde{\sigma}^2, \quad \text{and} \quad w_2 = 0.$$

As $(i\omega) \rightarrow 0$ the roots take the form

$$w_1 = 1 - \varepsilon^{1/2}\tilde{\sigma}(1 + \frac{\alpha_1}{\alpha_0}) - \tilde{\sigma}^2 + A(i\omega)^{-1} + O(\omega^{-2}), \quad \text{and} \quad w_2 = B(i\omega) + O(\omega^{-1}). \quad (4.182d)$$

Substituting (4.182d) into (4.182c) we obtain

$$w_1 = 1 - \varepsilon^{1/2}\tilde{\sigma}(1 + \frac{\alpha_1}{\alpha_0}) - \tilde{\sigma}^2 - i\omega^{-1}\tilde{\sigma} \left\{ \frac{\varepsilon^{1/2}}{\alpha_0^2}(\alpha_1 - \alpha_0) - \frac{\tilde{\sigma}}{\alpha_0\tau[1 - \varepsilon^{1/2}\tilde{\sigma}(1 + \frac{\alpha_1}{\alpha_0}) - \tilde{\sigma}^2]} \right\} + O(\omega^{-2}),$$

and

$$w_2 = \frac{-i\omega\alpha_0\tau}{(\alpha_0 + \tau)} \left\{ 1 - \varepsilon^{1/2}\tilde{\sigma}(1 + \frac{\alpha_1}{\alpha_0}) - \tilde{\sigma}^2 \right\} + O(\omega^{-1}).$$

If $\tilde{\sigma} > \tilde{\sigma}_c$ $\text{Im } w_1(\omega) < 0$ and $\text{Im } w_2(\omega) > 0$ so w_1 is stable and w_2 is unstable and if $\tilde{\sigma} < \tilde{\sigma}_c$ we cannot tell the sign of $\text{Im } w_1(\omega)$ but it is clear that $\text{Im } w_2(\omega) < 0$ so w_2 is stable. If $\tilde{\sigma} = \tilde{\sigma}_c$ the analysis is not valid and we return to the special case (4.183).

Recall the special cases.

Case 1: The purely thermal constraint viewed as the limit $\tilde{\sigma} \rightarrow 0$

We obtain

$$w_1 = i\omega\tilde{\sigma}^2[(1 - i\omega\tau)(1 - i\omega\alpha_0)]^{-1} + O(\tilde{\sigma}^3), \quad (4.184)$$

$$w_2 = 1 - \varepsilon^{1/2}\tilde{\sigma}(1 + \frac{1 - i\omega\alpha_1}{1 - i\omega\alpha_0}) - \tilde{\sigma}^2 - i\omega\tilde{\sigma}^2[(1 - i\omega\tau)(1 - i\omega\alpha_0)]^{-1} + O(\tilde{\sigma}^3). \quad (4.185)$$

Case 2: The purely mechanical constraint viewed as the limit $\tilde{\sigma} \rightarrow \infty$

We obtain

$$w_1 = 1 - \varepsilon^{1/2}\tilde{\sigma}(1 + \frac{1 - i\omega\alpha_1}{1 - i\omega\alpha_0}) - \tilde{\sigma}^2 + i\omega[(1 - i\omega\tau)(1 - i\omega\alpha_0)]^{-1} \left(1 - \varepsilon^{1/2}\tilde{\sigma}^{-1}(1 + \frac{1 - i\omega\alpha_1}{1 - i\omega\alpha_0})\right) + O(\tilde{\sigma}^{-2}), \quad (4.186)$$

$$w_2 = -i\omega[(1 - i\omega\tau)(1 - i\omega\alpha_0)]^{-1} \left(1 - \varepsilon^{1/2}\tilde{\sigma}^{-1}(1 + \frac{1 - i\omega\alpha_1}{1 - i\omega\alpha_0})\right) + O(\tilde{\sigma}^{-2}). \quad (4.187)$$

Numerical results

In Figure 4.10 illustrates two longitudinal waves are plotted for various values of $\tilde{\sigma}$. The branch $w_1(\omega)$ starting from the point $1 - 2\varepsilon^{1/2}\tilde{\sigma} - \tilde{\sigma}^2$ and ending at the point $1 - \varepsilon^{1/2}\tilde{\sigma}(1 + \alpha_1/\alpha_0) - \tilde{\sigma}^2$ and $w_2(\omega)$ starting and ending at the origin. In the low frequency $w_1(\omega)$ is stable and $w_2(\omega)$ is unstable and the situation reversed in the high frequency.

Varying the parameters $\alpha_1, \alpha_0, \varepsilon$ and τ while changing the magnitude of ω does not have any substantive influence on the stability.

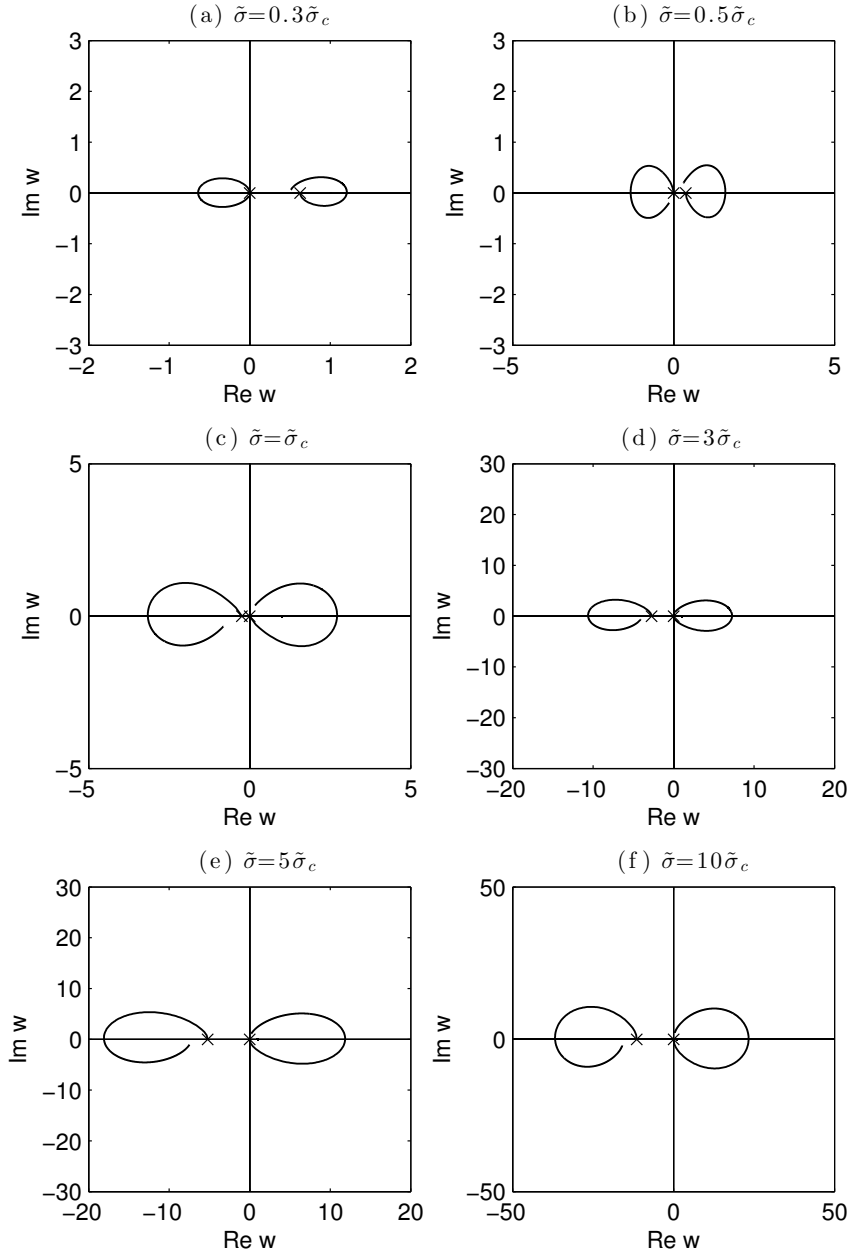


Figure 4.10: The longitudinal squared wave speeds of isotropic thermelastic material for TRDTE+GTE (2) with incompressibility at uniform temperature. For each part ($\varepsilon = 1, \alpha_0 = 0.01, \alpha_1 = 0.02, \tau = 0.01$), (a) $\tilde{\sigma} = 0.3\tilde{\sigma}_c$, (b) $\tilde{\sigma} = 0.5\tilde{\sigma}_c$, (c) $\tilde{\sigma} = \tilde{\sigma}_c$, (d) $\tilde{\sigma} = 3\tilde{\sigma}_c$, (e) $\tilde{\sigma} = 5\tilde{\sigma}_c$, (f) $\tilde{\sigma} = 10\tilde{\sigma}_c$.

Chapter 5

Concluding Remarks

We have shown that on taking into account the conditions $\alpha_1 > \alpha_0 > 0$ and $\tau > 0$ in unconstrained anisotropic thermoelastic materials, there are four waves which may propagate in each direction and all of them are stable in the context of TRDTE. However, in the context of TRDTE+GTE (1), the ad hoc theory of Chandrasekharaiah and Keshevan [23], one mode is infinite and stable in low and high frequencies limits but the other branches are finite, stable in the low frequency limits but unstable in the high frequency limits, increasing τ in this case does not affect the stability. In the context of TRDTE+GTE (2), Ignaczak's [25] more rational theory, all waves are finite, stable in the low frequency limits and unstable in the high frequency limits but three of them become stable in low and high frequencies by increasing τ .

When either the usual or the alternative deformation temperature constraint operates in the context of any of these theories, there is always one unstable wave and one which tends to infinity and the other branches are stable but by increasing α_1 and α_0 all of them become unstable.

In unconstrained isotropic materials there are two longitudinal waves propagating in each direction. Both of them are finite and stable in the context of TRDTE, but in the context of TRDTE+GTE (1) one mode is stable and the other is stable in the low frequency limit but unstable in the high frequency limits, increasing ε in this case does not affect the stability. In the context of TRDTE+GTE (2) both modes are

stable in the low frequency limit and unstable in the high frequency limit but with increasing ε one become stable in low and high frequencies and the other maintains the stability in the low frequency limit and instability in the high frequency limit.

If the usual deformation temperature constraint operates there are two longitudinal waves travelling in each direction; one is finite and the other tends to infinity. Both modes are unstable but with increasing $\tilde{\sigma}$ one mode maintains instability and the other becomes stable in the context of TRDTE and TRDTE+GTE (1). But in the context of TRDTE+GTE (2) one mode is unstable and the other is stable in the low frequency limits and the other is unstable in the high frequency limits, increasing $\tilde{\sigma}$ in this case does not affect the stability.

If the alternative deformation temperature constraint operates there are two longitudinal waves propagating in each direction; both of them are finite but one is stable and the other is unstable in the context of TRDTE and TRDTE+GTE (2). But in the context of TRDTE+GTE (1) one mode is unstable and tends to infinity and the other is finite, stable in the low frequency and unstable in the high frequency limits.

The results obtained for an anisotropic thermoelastic material of TRDTE and TRDTE + GTE(1) type which is constrained by the usual deformation temperature constraint when α_1 and α_0 are small enough, see Figures 2.6 and 3.6, are quite similar to those for an anisotropic thermoelastic GTE material which is constrained by the usual deformation temperature constraint, see Leslie and Scott [27, Figure 1].

The results obtained for an isotropic thermoelastic TRDTE material which is unconstrained, see Figure 2.4, are quite similar to those for an isotropic thermoelastic GTE material which is unconstrained, see Leslie and Scott [29, Figure 1].

Similarly, the results obtained for an anisotropic thermoelastic TRDTE material which is constrained by the usual deformation temperature constraint, see Figure 2.6, are similar to those for an anisotropic thermoelastic material TRDTE+GTE (1) which is also constrained by the usual deformation temperature constraint, see Figure 3.6.

The results obtained for an anisotropic thermoelastic material of TRDTE+GTE (1) type which is constrained by the alternative deformation temperature constraint, see

Figure 3.9, are similar to those for an anisotropic thermoelastic material of TRDTE+GTE (2) type which is constrained by the usual deformation temperature constraint, see Figure 4.3.

We have found that the results obtained for isotropic thermoelastic TRDTE materials which are constrained by the usual deformation temperature constraint, see Figure 2.12, are exactly the same as those for isotropic thermoelastic TRDTE+GTE (1) materials which are constrained by the usual deformation temperature constraint, see Figure 3.12 as $\tilde{\sigma} \geq \tilde{\sigma}_c$.

The results obtained for isotropic thermoelastic materials of TRDTE+GTE (1) type which are constrained by the alternative deformation temperature constraint, see Figure 3.14, are exactly the same as those for isotropic thermoelastic materials of TRDTE+GTE (2) type which are constrained by the usual deformation temperature constraint, see Figure 4.9 as $\tilde{\sigma} \geq \tilde{\sigma}_c$.

In conclusion, we have seen that instabilities are associated with the occurrence of the three relaxation times α_0, α_1 and τ even when the materials are unconstrained, except that in the TRDTE case there is no instability. There is another reason for these instabilities and that is the presence of a constraint of deformation-temperature type, either the usual or the alternative form.

The undesirable effects of instabilities may possibly be circumvented by assuming the constraints to hold only approximately or by using a new theory of deformation-entropy constraints.

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